Homework 6: Monotonic Sequences, Series

Assignments should be ***stapled*** *and written clearly and legibly.*

1. Prove that the sequence (a_n) converges, where $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

2. In this problem, we give an algorithm for computing $\sqrt{2}$. Let $a_1 = 2$, and define

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right), \text{ for } n \geq 1. \tag{1}$$

   (a) Prove that $a_n^2 \geq 2$ for all n. (Use proof by induction.)

   (b) Use part (a) and equation (1) to prove that $a_n - a_{n+1} \geq 0$ for all n.

   (c) Conclude that the sequence (a_n) converges.

   (d) Prove that $\lim_{n\to\infty} a_n = \sqrt{2}$.

   (e) Modify the sequence (a_n) so that it converges to $\sqrt{c}$. No formal proof is required for this part, but you should give a brief justification.

3. (a) Prove that if $0 < a < 2$, then $a < \sqrt{2a} < 2$.

   (b) Use part (a) to prove that the sequence

$$\left(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots\right)$$

   converges.

   (c) Find the limit.

4. §8.1, #8, 10.