

## Homework 5: More Limits

Assignments should be **stapled** and written clearly and legibly. Problems 6 and 7 are optional.

1. Let  $(a_n)$  be a convergent sequence. Suppose that  $\lim a_n > 0$ . Use the definition of a limit to prove that there exists  $N \in \mathbb{N}$  such that  $a_n > 0$  for all  $n \geq N$ .
2. From any given sequence  $(a_n)$  we can form the related sequence  $(b_n) = (5a_n + 2)$ . Use the definition of convergence of a sequence to prove that if  $(a_n)$  converges to 20, then  $(b_n)$  converges to \_\_\_\_\_. (First fill in the blank.)
3. Let  $(a_n)$  and  $(b_n)$  be sequences. Suppose that  $(a_n)$  converges to 0.
  - (a) Using the definition of convergence, prove that if  $(b_n)$  is bounded, then the sequence  $(a_n b_n)$  converges. (Note that you may not assume that  $(b_n)$  converges.)
  - (b) If the sequence  $(b_n)$  is not bounded, must the sequence  $(a_n b_n)$  necessarily converge? If so, prove it. If not, give a counterexample.
4. Give an example of a sequence  $(a_n)$  such that
  - (a)  $(a_n)$  converges to 0, but  $a_n \neq 0$  for all  $n$ .
  - (b)  $(a_n)$  is bounded but does not converge.
  - (c)  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$  but  $(a_n)$  does not converge.
  - (d)  $(|a_n|)$  converges but  $(a_n)$  does not.
5. Give examples of the following:
  - (a) two divergent sequences  $(a_n)$  and  $(b_n)$  for which  $(a_n + b_n)$  converges.
  - (b) two divergent sequences  $(c_n)$  and  $(d_n)$  for which  $(c_n d_n)$  converges.
6. (Challenging) Suppose that  $(a_n)$  is a convergent sequence and  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a bijection. Determine whether  $(a_{f(n)})$  converges. Prove your answer.
7. (More Challenging)
  - (a) Prove that if a sequence  $(a_n)$  is convergent, then the sequence of averages

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

is also convergent, and converges to the same limit.

- (b) Show by example that it is possible for a sequence to diverge but its sequence of averages to converge.