Homework 5: More Limits

*Assignments should be **stapled** and written clearly and legibly. Problems 6 and 7 are optional.*

1. Let (a_n) be a convergent sequence. Suppose that $\lim a_n > 0$. Use the definition of a limit to prove that there exists $N \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N$.

2. From any given sequence (a_n) we can form the related sequence $(b_n) = (5a_n + 2)$. Use the definition of convergence of a sequence to prove that if (a_n) converges to 20, then (b_n) converges to __________. (First fill in the blank.)

3. Let (a_n) and (b_n) be sequences. Suppose that (a_n) converges to 0.
   (a) Using the definition of convergence, prove that if (b_n) is bounded, then the sequence $(a_n b_n)$ converges. (Note that you may not assume that (b_n) converges.)
   (b) If the sequence (b_n) is not bounded, must the sequence $(a_n b_n)$ necessarily converge? If so, prove it. If not, give a counterexample.

4. Give an example of a sequence (a_n) such that
   (a) (a_n) converges to 0, but $a_n \neq 0$ for all n.
   (b) (a_n) is bounded but does not converge.
   (c) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ but (a_n) does not converge.
   (d) $(|a_n|)$ converges but (a_n) does not.

5. Give examples of the following:
   (a) two divergent sequences (a_n) and (b_n) for which $(a_n + b_n)$ coverges.
   (b) two divergent sequences (c_n) and (d_n) for which $(c_n d_n)$ converges.

6. (Challenging) Suppose that (a_n) is a convergent sequence and $f : \mathbb{N} \to \mathbb{N}$ is a bijection. Determine whether $(a_{f(n)})$ converges. Prove your answer.

7. (More Challenging)
   (a) Prove that if a sequence (a_n) is convergent, then the sequence of averages

   $$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

   is also convergent, and converges to the same limit.
   (b) Show by example that it is possible for a sequence to diverge but its sequence of averages to converge.