Homework 11: The Definition of Derivative

Directions. Assignments should be ***stapled*** *and written clearly and legibly. For problems 1 - 4, I recommend not using $\epsilon - \delta$ arguments. Instead you should use limit laws where appropriate.*

1. §6.1, #9.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying (i) $f(0) = 0$, (ii) $f'(0) = 3$, and (iii) $g(0) = 2$. Prove that fg is differentiable at 0, and find $(fg)'(0)$.

   Note: The product rule for derivatives cannot be used for this problem, since g may not be differentiable at 0. You must use the definition of derivative.

3. Let $f : (-1, 1) \to \mathbb{R}$ be a bounded function. In Homework 11, you proved that the function $g : (-1, 1) \to \mathbb{R}$ defined by $g(x) = xf(x)$ is continuous at $x = 0$. Use this result to prove that the function $h : (-1, 1) \to \mathbb{R}$ defined by $h(x) = x^2 f(x)$ is differentiable at 0, and find $h'(0)$.

   Note: The product rule for derivatives cannot be applied here either.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $\lim_{x \to 0} \frac{f(x)}{x}$ exists.
   (a) Prove that $\lim_{x \to 0} f(x)$ exists and find its value.
   (b) Prove that if $f(0) = 0$, then f is differentiable at 0.

5. Determine whether the following function is differentiable at 0. Prove your answer.

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$$