Homework 10: Continuity and the Intermediate Value Theorem

*Assignments should be **stapled** and written clearly and legibly.*

1. §5.3, #5, 7.

2. Let f be continuous on $[0,1]$ with $f(0) = f(1)$. Prove that there exists $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.

3. Prove that there exists a real number x such that
$$x^{177} + \frac{165}{1 + x^8 + \sin^2 x} = 125.$$

4. Suppose that f is continuous on $[a,b]$ and for every $x \in [a,b]$, $f(x) \in \mathbb{Q}$. Prove that f is constant on $[a,b]$. (Hint: use proof by contradiction.)

5. Let $f \colon \mathbb{R} \to \mathbb{R}$ be continuous.

   (a) Suppose that $D \subseteq \mathbb{R}$ is dense in $\mathbb{R}$, i.e., $\overline{D} = \mathbb{R}$. Prove that if $f(x) = 0$ for all $x \in D$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

   (b) Use part (a) and Problem 1 of Homework 8 to prove that if $f(x) = 0$ for all $x \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.