Homework 14: Covariance

1. Suppose that X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{4}{5}(xy+1), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

   Find $\text{Cov}(X,Y)$, the covariance of X and Y.

2. Let X be uniformly distributed on $[-1,1]$, and let $Y = X^2$. Show that $\text{Cov}(X,Y) = 0$ even though X and Y are far from being indepedent: given the value of X, $Y = X^2$ is completely determined. Prove that X and Y are not independent.

3. Consider random variables $X_1, \ldots, X_n$ with

$$E(X_i) = 3 \text{ for all } i$$
$$E(X_i X_j) = 12 \text{ for } i \neq j$$
$$E(X_i^2) = 11 \text{ for all } i$$

   Find $\text{Var}(X_1 + \cdots + X_n)$.

4. Let X and Y be independent, each with mean 2 and variance 1. Let $U = 3X + 2Y$, $V = 2X - 3Y$. Find $\text{Var}\, U$ and $\text{Cov}(U,V)$.

5. Suppose that when they arrive at a party, n people throw their hats into a closet. Upon leaving, each person takes a random hat from the closet. Let X be the number of people who leave the party with their own hats. Find $E(X)$ and $\text{Var}(X)$.

   Hint: Use indicators. Let I_j be the indicator of the event "person j leaves with her own hat".

6. A population of n people vote in an election. d vote democratic and $n-d$ vote republican. In the next election, the probability of a democratic voter switching to republican is p_1, and the probability of a republican voter switching to democratic is p_2. Let X be the number of democratic votes in the second election. Find $E(X)$ and $\text{Var}(X)$.

   Hint: Use indicators.