Homework 7: Basis

*Instructions. Assignments should be **stapled** and written clearly and legibly. All answers must be justified. Problem 5 is optional.*

1. §4.4, #13(b), 14(b), 24(a)(b), 27(a)(b).

2. Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space V.
   (a) Suppose that $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$, and $\mathbf{v}$ is not a scalar multiple of $\mathbf{u}$. Is $\{\mathbf{u}, \mathbf{v}\}$ necessarily linearly independent? If so, prove it. If not, give a concrete counterexample.
   (b) If it is only known that $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$, is $\{\mathbf{u}, \mathbf{v}\}$ necessarily linearly independent? If so, prove it. If not, give a concrete counterexample.
   (c) Suppose that none of the three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ is a scalar multiple of either of the other two vectors. In other words, $\mathbf{u}_1$ is not a scalar multiple of $\mathbf{u}_2$ or of $\mathbf{u}_3$; $\mathbf{u}_2$ is not a scalar multiple of $\mathbf{u}_1$ or of $\mathbf{u}_3$; and $\mathbf{u}_3$ is not a scalar multiple of $\mathbf{u}_1$ or of $\mathbf{u}_2$. Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ necessarily linearly independent? If so, prove it. If not, give a concrete counterexample.

3. Suppose that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly idependent set of vectors in a vector space V. Prove that
   (a) $\operatorname{Span} \mathcal{B}$ is a subspace of V.
   (b) $\mathcal{B}$ is a basis for $\operatorname{Span} \mathcal{B}$.

4. In this problem you will prove that $\{e^x, e^{2x}\}$ is a basis for $\operatorname{Span}\{e^x, e^{2x}\}$.
   (a) Prove that $\{e^x, e^{2x}\}$ is linearly independent in $F(-\infty, \infty)$.
   (b) Prove that $\operatorname{Span}\{e^x, e^{2x}\}$ is a subspace of $F(-\infty, \infty)$.
   (c) Use Problem 3 to prove that $\{e^x, e^{2x}\}$ is a basis for $\operatorname{Span}\{e^x, e^{2x}\}$.

5. Let p, q, r and s be polynomials of degree at most 3. Which, if any, of the following two conditions is sufficient for the conclusion that the polynomials are linearly independent?
   (a) At 1 each polynomial has the value 0.
   (b) At 0 each polynomial has the value 1.