Homework 4: Subspace, Linear Combination, Span

Instructions. *All answers to textbook problems should be justified. For problems 5 and 6 (except 6(c)), you may not Theorems 1.2 or 1.3 from the study guide. Assignments should be* ***stapled*** *and written clearly and legibly. Problem 7 is optional.*

1. §4.2, #3(a), #5(a)(b).

2. Give an example of a nonempty subset of $M_{2,3}$ which is not a subspace of $M_{2,3}$.

3. Consider the set W of all vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} a \\ b \\ -2b \\ a \end{bmatrix}$.

   (a) Prove that W is a subspace of $\mathbb{R}^4$ using Theorem 1.2 from the study guide.

   (b) Prove that W is a subspace of $\mathbb{R}^4$ using Theorem 1.3 from the study guide.

4. §4.2, #10(b)(c), 12(a)(b), 14.

5. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in a vector space V. Prove that $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is a subspace of $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   Hint. Begin the proof as follows: "Suppose that $\mathbf{z}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$. I must show that $\mathbf{z}$ is also in $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$."

6. Let $\mathbf{u}, \mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ be vectors in a vector space V. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are in $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ and c is a scalar.

   (a) Prove that $\mathbf{x} + \mathbf{y}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   (b) Prove that $c\mathbf{x}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   (c) What do parts (a) and (b) tell you about $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$? (You may use any theorem to answer this question.)

   Hint. The proofs of (a) and (b) are taken from the proof of a theorem from class.

7. (Putnam Competition) Let S be a set and let $\circ$ be a binary operation on S satisfying the two laws

   $$x \circ x = x \text{ for all } x \text{ in } S, \text{ and}$$
   $$(x \circ y) \circ z = (y \circ z) \circ x \text{ for all } x, y, z \text{ in } S.$$

   Show that $\circ$ is associative and commutative.