Homework 3: Vector Spaces

Instructions. Assignments should be **stapled** and written clearly and legibly. Problems 4 and 5 are optional.

1. §4.1, #2, 7, TF (True-False Exercises).

2. Let V be the set of all matrices of real numbers with one column and two rows, with addition and scalar multiplication defined as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2u_1v_1 \\ u_2 + v_2 \end{bmatrix} \qquad \text{and} \qquad c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 5cu_2 \\ 5cu_1 \end{bmatrix}$$

   With this addition and scalar multiplication, V is not a vector space. Identify all vector space axioms which fail, and briefly explain why each of these axioms fail (in most cases, a counterexample will be sufficient). If axiom 3 holds, then give an explicit additive identity.

3. Let V be a vector space. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in V, and let b and c be scalars. Using only the definition of a vector space, prove
   (a) $c\,\mathbf{0} = \mathbf{0}$
   (b) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{v}+(\mathbf{w}+\mathbf{u})$ (Hint: use only the commutative and associative axioms.)
   (c) $(b+c)(\mathbf{u}+\mathbf{v}) = (c\mathbf{u}+c\mathbf{v}) + (b\mathbf{u}+b\mathbf{v})$

4. (Challenge) Prove that there does not exist a vector space which contains exactly two elements.

5. (Challenge) Consider the following infinite system of linear equations in an infinite number of variables $x_1, x_2, x_3, \ldots$:

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 + x_4 + x_6 &= 0 \\ x_3 + x_5 + x_7 &= 0 \\ \vdots \quad \vdots \quad \vdots & \end{aligned}$$

   Determine all solutions. Determine the number of free variables and the number of leading variables.