Homework 14: The Matrix of a Linear Transformation

1. §1.5, #9, 11(a), 16.

2. In this problem you will prove the following trigonometric identities:

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \end{aligned} \tag{1}$$

   Use the following strategy. Let T be counterclockwise rotation by x and S counterclockwise rotation by y (both in $\mathbb{R}^2$). Find the standard matrix for $S \circ T$ in two ways: (i) by multiplying the standard matrices for S and T, and (ii) by observing that $S \circ T$ is rotation by $x + y$. The matrices obtained in (i) and (ii) must be equal, so their entries must be equal.

3. Use the trigonometric identities (1) to find formulas for $\cos(2x)$ and $\sin(2x)$.

4. In an earlier homework problem, you considered the linear transformation $T : P_2 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(-1) \\ p(1) \end{bmatrix}$. Determine whether T is onto. **Prove** your answer.

5. Construct the following:
   (a) A matrix A such that the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one but not onto.
   (b) A matrix B such that the matrix transformation $T(\mathbf{x}) = B\mathbf{x}$ is onto but not one-to-one.
   (c) A matrix C such that the matrix transformation $T(\mathbf{x}) = C\mathbf{x}$ is one-to-one and onto.

   Briefly explain why your given matrices satify the required properties.

6. (Optional) Let $T : V \to W$ be linear. A **left inverse** of T is a linear transformation $L : W \to V$ such that $L \circ T = I_V$, and a **right inverse** of T is a linear transformation $R : W \to V$ such that $T \circ R = I_W$.
   (a) Prove that if T has a left inverse, then T is one-to-one.
   (b) Prove that if T has a right inverse, then T is onto.
   (c) Prove that if T has a left inverse L and a right inverse R, then T is an isomorphism and $L = R$.
   (d) Give an example of a linear transformation T that has a left inverse but not a right inverse.
   (e) Give an example of a linear transformation T that has a right inverse but not a left inverse.

7. §8.4, #8.

8. Let $T : V \to W$ be linear, and let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis for V and $\mathcal{B}' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ a basis for W. Suppose that $T(\mathbf{u}_1) = 3\mathbf{w}_1 + 5\mathbf{w}_2 - 7\mathbf{w}_3$ and $T(\mathbf{u}_2) = 2\mathbf{w}_1 + 4\mathbf{w}_3$.

   (a) Find $[T]_{\mathcal{B}',\mathcal{B}}$.

   (b) If $[\mathbf{x}]_{\mathcal{B}} = \left[\begin{smallmatrix} 7 \\ -2 \end{smallmatrix}\right]$, then what is $[T(\mathbf{x})]_{\mathcal{B}'}$?

9. Let $\mathcal{B}$ be the standard basis of $\mathbb{R}^2$, let $\mathcal{C}$ be the basis of $\mathbb{R}^3$ consisting of the three vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and let $\mathcal{D}$ be the standard basis of $\mathbb{R}^3$. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T\left(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\right) = a\mathbf{u}_1 + b\mathbf{u}_2 + (a+b)\mathbf{u}_3$. Find $[T]_{\mathcal{C},\mathcal{B}}$ and $[T]_{\mathcal{D},\mathcal{B}}$.