Homework 10: Kernel, Range

*Assignments should be **stapled** and written clearly and legibly. Problems 4, 5, and 6 are optional.*

1. §8.1, #10, 11, 24.

2. Let V and W be vector spaces, and let $T : V \to W$ be linear. Prove that if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ spans V, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ spans $R(T)$.

3. Let A be an $m \times n$ matrix. Prove that $\operatorname{Nul} A = \{\mathbf{0}\}$ if and only if the columns of A are linearly independent.

4. Let $T : V \to W$ be a linear transformation. Let w be an element of W, and let v_0 be an element of V which T maps to w. Prove that the set of all vectors which T maps to w is equal to $\{v_0 + u : u \in \ker T\}$. (Note that this is not, in general, a subspace of V – why not?)

5. (Differential Equations) Let $V = W = C^\infty(-\infty, \infty)$ be the vector space of infinitely differentiable functions. Let $T : V \to W$ be the linear transformation

$$T(f) = a_m f^{(m)} + a_{m-1} f^{(m-1)} + \cdots + a_1 f,$$

   where $f^{(i)}$ denotes the i-th derivative of f, and $a_m, a_{m-1}, \dots, a_1$ are constants.

   (a) Recall that $\ker T = \{f \in V : T(f) = 0\}$. In terms of differential equations, what does $\ker T$ represent?

   (b) Let $g \in W$, and let $H = \{h \in V : T(h) = g\}$. In terms of differential equations, what does H represent?

   (c) Let h_0 be any element of H. Use Problem 3 to prove that $H = \{h_0 + f : f \in \ker T\}$.

6. Let $T : V \to V$ be a linear transformation, and let $\mathbf{x}$ be a vector in V. Suppose that, for some positive integer m, $T^{m-1}(\mathbf{x}) \neq \mathbf{0}$ but $T^m(\mathbf{x}) = \mathbf{0}$. Prove that $\{\mathbf{x}, T(\mathbf{x}), \dots, T^{m-1}(\mathbf{x})\}$ is linearly independent.