DEPARTMENT OF MATHEMATICS AND PHYSICS
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DEFINITIONS & THEOREMS

STUDY GUIDE FOR A COURSE IN LINEAR ALGEBRA

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Contents

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Unit 1: Vector Space, Subspace, Linear Combination, Span, Linear Independence, Basis

Definition 1.1: Binary and Scalar Operations

(i) A **binary operation** on a set V is a rule which, for any two elements u and v in V, produces a third element in V. (Produced element sometimes denoted by $u+v$, $u \oplus v$, $u \cdot v$, or uv.)

(ii) A **scalar operation** on a set V is a rule which, for any real number k and any element u in V, produces an element of V. (Produced element sometimes denoted by $k \cdot v$ or kv.)

Definition 1.2: Vector Space

A vector space consists of the following:

- A set V
- A binary opertion on V (called addition, denoted $+$)
- A scalar operation on V (called scalar multiplication, denoted $\cdot$)

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and k, m in $\mathbb{R}$,

(i) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) There exists an element in V, denoted by $\mathbf{0}$, such that $\mathbf{0}+\mathbf{u} = \mathbf{u}$ for every $\mathbf{u}$ in V.

(iv) For every $\mathbf{u}$ in V, there exists an element $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$.

(v) $k \cdot (\mathbf{u}+\mathbf{v}) = k \cdot \mathbf{u} + k \cdot \mathbf{v}$

(vi) $(k+m) \cdot \mathbf{u} = k \cdot \mathbf{u} + m \cdot \mathbf{v}$

(vii) $k \cdot (m \cdot \mathbf{u}) = (km) \cdot \mathbf{u}$

(viii) $1 \cdot \mathbf{u} = \mathbf{u}$

Definition 1.3: Additive Identity and Additive Inverse

In Definition 1.2 of a vector space above, $\mathbf{0}$ is called the **additive identity** or the **zero vector** of V, and for $\mathbf{u}$ in V, $-\mathbf{u}$ is called the **additive inverse** of $\mathbf{u}$.

Definition 1.4: Subspace

A **subspace** of a vector space V is a subset W of V which is itself a vector space.

Definition 1.5: Linear Combination

A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a vector of the form $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r$, where $k_1, \ldots, k_r$ are scalars.

Definition 1.6: Span

The **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_r$, denoted $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_r$.

Definition 1.7: Linearly independent

Let V be a vector space and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in V. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is **linearly independent** if $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ implies $k_1 = \cdots = k_r = 0$.

Definition 1.8: Basis

A **basis** for a vector space V is a set of vectors $\mathscr{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that

(i) $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$, and

(ii) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

Definition 1.9: Coordinates

Let $\mathscr{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of a vector space V, and let $\mathbf{u}$ be a vector in V. Then $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ for unique scalars $c_1, \dots, c_n$, by Theorem 1.6. The scalars $c_1, \dots, c_n$ are called the **coordinates** of $\mathbf{u}$ relative to $\mathscr{B}$, and the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

denoted by $[\mathbf{u}]_{\mathscr{B}}$, is called the **coordinate vector** of $\mathbf{u}$ relative to $\mathscr{B}$.

Definition 1.10: Dimension

The **dimension** of a nonzero vector space V is the number of vectors in a basis for V.

Definition 1.11: Product of Matrix and Vector

Let $A = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ be an $m \times n$ matrix with column vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$. Then $A\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$.

Definition 1.12: Nullspace

Let A be an $m \times n$ matrix. The **nullspace** of A, denoted by $\mathrm{Nul}\,A$, is the set of all solutions to $A\mathbf{x} = \mathbf{0}$.

Definition 1.13: Column Space

Let A be an $m \times n$ matrix. The **column space** of A, denoted by $\mathrm{Col}\,A$, is the span of the column vectors of A.

Theorem 1.1

Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar. Then

(i) $0\mathbf{u} = \mathbf{0}$

(ii) $k\mathbf{0} = \mathbf{0}$

(iii) $(-1)\mathbf{u} = -\mathbf{u}$

Theorem 1.2

Let V be a vector space and let W be a subset of V. Then W is a subspace of V if:

(i) for every $\mathbf{u}$ and $\mathbf{v}$ in W, $\mathbf{u}+\mathbf{v}$ is in W (i.e., W is closed under vector addition); and

(ii) for every $\mathbf{u}$ in W and scalar k, $k\mathbf{u}$ is in W (i.e., W is closed under scalar multiplication); and

(iii) the zero vector of V lies in W.

Theorem 1.3

Let V be a vector space and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in V. Then $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a subspace of V.

Theorem 1.4

A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.

Theorem 1.5

Let $\mathscr{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.

(i) If any vector of V is added to $\mathscr{B}$, then $\mathscr{B}$ is no longer linearly independent.

(ii) If any vector is removed from $\mathscr{B}$, then $\mathscr{B}$ no longer spans V.

Theorem 1.6

Let $\mathscr{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ be a basis of a vector space V. Then every $\mathbf{u}$ in V can be written in exactly one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, that is, can be expressed as

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n,$$

for unique scalars $c_1, \ldots, c_n$.

Theorem 1.7

All bases of a vector space V have the same number of elements.

Theorem 1.8

In $\mathbb{R}^n$, the following have the same solutions:

(i) The vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{u}$.

(ii) The linear system of equations with augmented matrix $[\mathbf{v}_1 \cdots \mathbf{v}_p \mid \mathbf{u}]$.

(iii) The matrix equation $[\mathbf{v}_1 \cdots \mathbf{v}_p]\,\mathbf{x} = \mathbf{u}$.

Lemma 1.1

Let A be an $m \times n$ matrix, let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(i) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(ii) $A(c\mathbf{u}) = c(A\mathbf{u})$.

Theorem 1.9

Let A be an $m \times n$ matrix. Then $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

Theorem 1.10

Let A be a matrix with n columns. Then $\dim(\operatorname{Nul} A) + \dim(\operatorname{Col} A) = n$.

Unit 2: Introduction to Linear Transformations

Definition 2.1: Linear Transformation

Let V and W be vector spaces. A transformation (or mapping) $T : V \to W$ is **linear** if it satisfies the following conditions:

(i) For every $\mathbf{u}, \mathbf{v}$ in V, $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

(ii) For every $\mathbf{u}$ in V and scalar c, $T(c\mathbf{u}) = cT(\mathbf{u})$.

Theorem 2.1

Let $T : V \to W$ be linear. Then

(i) $T(\mathbf{0}) = \mathbf{0}$.

(ii) $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$, for any scalars $c_1, \ldots, c_p$ and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in V.

Definition 2.2: Matrix Transformation

A **matrix transformation** is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$, for some fixed $m \times n$ matrix A.

Theorem 2.2

A matrix transformation is linear.

Definition 2.3: Kernel and Range

Let $T : V \to W$ be linear. Then

(i) The **kernel** of T, denoted $\ker(T)$, is the set of vectors in V which T maps to $\mathbf{0}$.

(ii) The **range** of T, denoted $R(T)$, is the set of vectors in W which have at least one vector in V mapping to them.

Theorem 2.3

Let $T : V \to W$ be linear. Then $\ker(T)$ is a subspace of V and $R(T)$ is a subspace of W.

Theorem 2.4

Let A be an $m \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. Then $\ker(T) = \operatorname{Nul} A$ and $R(T) = \operatorname{Col} A$.

Theorem 2.5

Let $T : V \to W$ be linear. Then $\dim(\ker T) + \dim(R(T)) = \dim V$.

Theorem 2.6

Let $T : V \to W$ be linear. Then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Theorem 2.7

Let W be a subspace of V. If $\dim W = \dim V$, then $W = V$.

Theorem 2.8

Let $T : V \to W$ be linear, and suppose that $\dim V = \dim W$. Then T is one-to-one if and only if T is onto.

Definition 2.4: Composition

Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then the **composition** of S with T, denoted $S \circ T$, is the map from U to W defined by $(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$ for $\mathbf{u} \in U$.

Theorem 2.9

Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then the composition $S \circ T : U \to W$ is a linear transformation.

Definition 2.5: Identity Transformation

For any vector space V, the **identity transformation** $I : V \to V$ is defined by $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V.

Theorem 2.10

Let $T : V \to W$ be a linear transformation. Then $T \circ I = I \circ T = T$.

Definition 2.6: Inverse Transformation

Let $T : V \to W$ be one-to-one. Then there exists an **inverse transformation** $T^{-1} : R(T) \to V$ such that $T^{-1}(T(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$ in V.

Theorem 2.11

Let $T : V \to W$ be one-to-one. Then $T^{-1} \circ T = I$.

Definition 2.7: Isomorphism

An **isomorphism** is a bijective linear transformation.

Definition 2.8: Isomorphic

If $T : V \to W$ is an isomorphism, then V and W are said to **isomorphic**.

Theorem 2.12

If $T : V \to W$ is an isomorphism, then $\dim V = \dim W$.

Theorem 2.13

Suppose that V is a vector space and $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. Then the mapping $T : V \to \mathbb{R}^n$ given by $T(\mathbf{u}) = [\mathbf{u}]_B$ is an isomorphism.

Unit 3: The Matrix of a Linear Transformation

Theorem 3.1

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then for $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$, where A is the matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$. The matrix $A = [T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ is called the **standard matrix** for T.

Theorem 3.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then T is a linear transformation if and only if T is a matrix transformation.

Theorem 3.3

Suppose that the standard matrix for S is A and the standard matrix for T is B. Then the standard matrix for $S \circ T$ is AB.

Definition 3.1: Invertible and Inverse

Let A be an $n \times n$ matrix. Then A is said to be **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, B is called the **inverse** of A, and we write $B = A^{-1}$.

Theorem 3.4

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let A be the standard matrix for T. Then T is an isomorphism if and only if A is invertible. In this case, the standard matrix for T^{-1} is A^{-1}.

Theorem 3.5

Let A be an $n \times n$ matrix. Then A is invertible if and only if A can be row reduced to I_n.

Theorem 3.6

Let A be an $n \times n$ matrix, and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 3.7

Let A be an $n \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\mathbf{x}) = A\mathbf{x}$. The following are equivalent:

(i) T is an isomorphism.

(ii) T is one-to-one.

(iii) T is onto.

(iv) $\ker T = \{\mathbf{0}\}$.

(v) $R(T) = \mathbb{R}^n$.

(vi) A is invertible.

(vii) A row-reduces to I_n.

(viii) $\operatorname{Nul} A = \{\mathbf{0}\}$.

(ix) The columns of A are linearly independent.

(x) $\operatorname{Col} A = \mathbb{R}^n$.

(xi) The columns of A span $\mathbb{R}^n$.

Theorem 3.8

Let $T : V \to W$ be linear. Let $\mathscr{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis for V and $\mathscr{B}' = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ a basis for W. Then there exists a matrix $[T]_{\mathscr{B}',\mathscr{B}}$ such that for every $\mathbf{v}$ in V, $[T(\mathbf{v})]_{\mathscr{B}'} = [T]_{\mathscr{B}',\mathscr{B}} \cdot [\mathbf{v}]_{\mathscr{B}}$.

Theorem 3.9

Let $T : V \to W$ be linear. Let $\mathscr{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis for V and $\mathscr{B}' = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ a basis for W. Then

$$[T]_{\mathscr{B}',\mathscr{B}} = \begin{bmatrix} [T(\mathbf{u}_1)]_{\mathscr{B}'} \cdots [T(\mathbf{u}_n)]_{\mathscr{B}'} \end{bmatrix}$$

Theorem 3.10

Let $T : U \to V$ and $S : V \to W$ be linear. Let $\mathscr{B}, \mathscr{B}', \mathscr{B}''$ be bases for vector spaces U, V, W respectively. Then $[S \circ T]_{\mathscr{B}'',\mathscr{B}} = [S]_{\mathscr{B}'',\mathscr{B}'} \cdot [T]_{\mathscr{B}',\mathscr{B}}$.

Definition 3.2: Change of Coordinates Matrix

Let $\mathscr{B}, \mathscr{B}'$ be bases for a vector space V. Then $[I]_{\mathscr{B}',\mathscr{B}}$ is called the **change of coordinates matrix** from $\mathscr{B}$ to $\mathscr{B}'$ coordinates.

Theorem 3.11

Let $\mathscr{B}, \mathscr{B}'$ be bases for a vector space V. Then

(i) For any $\mathbf{v}$ in V, $[\mathbf{v}]_{\mathscr{B}'} = [I]_{\mathscr{B}',\mathscr{B}} \cdot [\mathbf{v}]_{\mathscr{B}}$.

(ii) $[I]_{\mathscr{B},\mathscr{B}} = I_n$, where $n = \dim V$.

(iii) $[I]_{\mathscr{B}',\mathscr{B}}$ is invertible.

(iv) $([I]_{\mathscr{B}',\mathscr{B}})^{-1} = [I]_{\mathscr{B},\mathscr{B}'}$.

Notation 3.1

$[T]_{\mathscr{B},\mathscr{B}}$ is often denoted by just $[T]_{\mathscr{B}}$.

Theorem 3.12

[Change of Basis Formula] Let $T : V \to V$ be a linear operator. Let $\mathscr{B}, \mathscr{B}'$ be bases for V. Then

$$[T]_{\mathscr{B}'} = [I]_{\mathscr{B}',\mathscr{B}} \cdot [T]_{\mathscr{B}} \cdot [I]_{\mathscr{B},\mathscr{B}'}$$

Unit 4: Inner Product Spaces

Definition 4.1: Inner Product Space

Let V be a vector space. An **inner product** on V is a rule which assigns to each pair of vectors $\mathbf{u}, \mathbf{v}$ in V a scalar, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars c,

(i) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

(ii) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

(iii) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

(iv) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$.

A vector space with an inner product is called an **inner product space**.

Definition 4.2: Length, Distance, Unit Vector

Let V be an inner product space.

(i) For $\mathbf{v}$ in V, the **norm** (or **length**) of $\mathbf{v}$ is defined by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

(ii) For $\mathbf{u}, \mathbf{v}$ in V, the **distance** between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.

(iii) A **unit vector** is a vector of norm 1.

(iv) The set of all unit vectors in V is called the **unit circle** of V.

Definition 4.3: Orthogonal (Two Vectors)

Let V be an inner product space. Vectors $\mathbf{u}$ and $\mathbf{v}$ in V are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Definition 4.4: Orthogonal and Orthonormal (Set of Vectors)

A set S of two or more vectors in an inner product space is said to be **orthogonal** if every two distinct vectors in S are orthogonal. The set S is **orthonormal** if S is orthogonal and consists entirely of unit vectors.

Definition 4.5: Orthogonal Complement

Let V be an inner product space and let W be a subspace of V. The **orthogonal complement** of W, denoted $W^{\perp}$, is the set of vectors of V which are orthogonal to all vectors in W.

Theorem 4.1

Let V be an inner product space. Then

(i) $\langle \mathbf{v}, \mathbf{0} \rangle = 0$ and $\langle \mathbf{0}, \mathbf{v} \rangle = 0$, for every $\mathbf{v}$ in V.

(ii) $\langle c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n, \mathbf{w} \rangle = c_1\langle \mathbf{v}_1, \mathbf{w} \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{w} \rangle$, for all scalars $c_1, \ldots, c_n$ and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}$.

Theorem 4.2

Let V be an inner product space. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in V and let c be a scalar. Then

(i) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.

(ii) $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

Theorem 4.3

Let V be an inner product space. Let $\mathbf{v}$ be in V and let c be a scalar. Then

(i) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.

(ii) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector, if $\mathbf{v} \neq \mathbf{0}$.

Theorem 4.4

If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Theorem 4.5

Let V be an inner product space, and let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for V. Then for $\mathbf{u}$ in V, $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, where

$$c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}, \quad \text{for } i = 1, \ldots, n.$$

Theorem 4.6

Let W be a subspace of an inner product space V. Then

(i) $W^\perp$ is a subspace of V; and

(ii) $W \cap W^\perp = \{\mathbf{0}\}$.

Theorem 4.7

Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space V with $\mathbf{v} \neq \mathbf{0}$. Let $L = \operatorname{span}\{\mathbf{v}\}$, a one-dimensional subspace of V. The we can uniquely write $\mathbf{u} = \mathbf{y} + \mathbf{z}$, with $\mathbf{y}$ in L and $\mathbf{z}$ in $L^\perp$. Explicitly,

$$\mathbf{y} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v}, \quad \text{and} \quad \mathbf{z} = \mathbf{u} - \mathbf{y}.$$

The vector $\mathbf{y}$ is called the **orthogonal projection of u onto** L and denoted by $\operatorname{proj}_L \mathbf{u}$ or $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

Theorem 4.8

Let $\mathbf{u}$ be a nonzero vector in an inner product space V, and let W be a finite dimensional subspace of V. Then we can uniquely write $\mathbf{u} = \mathbf{y} + \mathbf{z}$, with $\mathbf{y}$ in W and $\mathbf{z}$ in $W^\perp$. The vector $\mathbf{y}$ is called the **orthogonal projection of u onto** W and denoted by $\operatorname{proj}_W \mathbf{u}$, and $\mathbf{z}$ is called the **component of u orthogonal to** W.

Unit 5: Determinants

Theorem 5.1

Let A be a square matrix.

(i) If two rows of A are interchanged to produce a matrix B, then $\det B = -\det A$.

(ii) If one row of A is multiplied by a constant k to produce B, then $\det B = k \det A$.

(iii) If a multiple of one row of A is added to another row to produce B, then $\det B = \det A$.

Theorem 5.2

Let A be a square matrix. Then A is invertible if and only if $\det A \neq 0$.

Unit 6: Eigenvectors and Eigenvalues

Definition 6.1: Eigenvector, Eigenvalue

Let A be an $n \times n$ matrix. An **eigenvector** of A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda \mathbf{x}$ for some scalar λ. The scalar λ is called the **eigenvalue** corresponding to $\mathbf{v}$.

Theorem 6.1

Let A be an $n \times n$ matrix. Then λ is an eigenvalue of A if and only if $\det(\lambda I_n - A) = 0$.

Definition 6.2: Characteristic Polynomial

Let A be an $n \times n$ matrix. The **characteristic polynomial** of A is $\det(\lambda I_n - A)$.

Theorem 6.2

Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. Then $\mathbf{x}$ is an eigenvector of A corresponding to λ if and only if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x}$ is in $\operatorname{Nul}(\lambda I_n - A)$.

Definition 6.3: Eigenspace

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Then $\operatorname{Nul}(\lambda I_n - A)$ is called the **eigenspace** of A corresponding to λ (or sometimes just the **λ-eigenspace** of A).

Theorem 6.3

Let A be an $n \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. Suppose that $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ consisting of eigenvectors for A (i.e., an **eigenbasis** for A). Suppose that the eigenvalues of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are $\lambda_1, \ldots, \lambda_n$. Then $[T]_B$ is the following diagonal matrix:

$$[T]_B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

If B' is the standard basis for $\mathbb{R}^n$, then by the change of basis theorem, $[T]_{B'} = [I]_{B',B}[T]_B[I]_{B,B'}$. This is often written $A = PDP^{-1}$.

Definition 6.4: Diagonalizable

An $n \times n$ matrix is said to be **diagonalizable** if it has an eigenbasis, i.e., a basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Theorem 6.4

Let A be an $n \times n$ matrix. If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A, and if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are corresponding eigenvectors, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

Definition 6.5: Algebraic and Geometric Multiplicities

Let λ be an eigenvalue of A.

(i) The **algebraic multiplicity** of λ is the multiplicity of A as a zero of the characteristic polynomial of A.

(ii) The **geometric multiplicity** of λ is the dimension of the λ eigenspace of A.

Theorem 6.5

Let A be an $n \times n$ matrix, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A.

(i) The geometric multiplicity of any eigenvalue is less than or equal to its algebraic multiplicity.

(ii) A is diagonalizable if and only if the geometric multiplicity of each eigenvalue is equal to its algebraic multplicity.