# ON THE TENSOR PRODUCT OF TWO BASIC REPRESENTATIONS OF $U_v(\hat{sl}_e)$

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Abstract. Let $\{B(\Lambda_m) \mid m \in \mathbb{Z}/e\mathbb{Z}\}$ be the set of level one $\mathfrak{g}(A^{(1)}_{e-1})$-crystals, and consider the realization of $B(\Lambda_m)$ using $e$-restricted partitions. We prove a purely Young diagrammatic criterion for an element of $B(\Lambda_0)^{\otimes d_1} \otimes B(\Lambda_m)^{\otimes d_2}$ to be in the component $B(d_1\Lambda_0 + d_2\Lambda_m)$. As an application, we give a non-recursive characterization of simple modules of the Hecke algebra of type $B$. In the course of the proof, we also obtain a combinatorial description of the second type of Kashiwara's Demazure crystal in $B(\Lambda_m)$.

## 1. Introduction

Let $\mathcal{H}_n(Q, q)$ be the Hecke algebra of type $B$ defined over an algebraically closed field $F$ of characteristic $\ell$. The $F$-algebra $\mathcal{H}_n(Q, q)$ is generated by $T_0, \dots, T_{n-1}$ subject to the quadratic relations $(T_0 - Q)(T_0 + 1) = 0$, $(T_i - q)(T_i + 1) = 0$, for $1 \le i < n$, and the type $B$ braid relations. Let $q$ be a power of a prime $p \neq \ell$. Motivated by a desire to generalize their famous work on the classification of simple $FGL_n(q)$-modules to other classical groups, Dipper and James initiated the study of modular representations of Hecke algebras of type $B$, where $q$ is an arbitrary element in $F$. They proved a certain Morita equivalence theorem [DJ, Theorem 4.14] and as a result, they classified simple $\mathcal{H}_n(Q, q)$-modules in the case when $-Q \notin q^{\mathbb{Z}}$ [DJ, Theorem 5.6]. Suppose that $q \neq 1$ and $-Q \in q^{\mathbb{Z}}$. Then the classification of simple $\mathcal{H}_n(Q, q)$-modules was achieved in [A2, Theorem 4.2], which completed the previous work [AM]. The classification is given for cyclotomic Hecke algebras associated with $G(r, 1, n)$, which is defined by replacing $(T_0 - Q)(T_0 + 1) = 0$ with $(T_0 - v_1) \cdots (T_0 - v_r) = 0$ in the above definition.[1] We note here that Geck-Rouquier theory provides us with another approach for classifying simple $\mathcal{H}_n(Q, q)$-modules.[2] The advantage of their approach is that it works for arbitrary finite Hecke algebras. It is also worth mentioning that Jacon generalized the theory to cyclotomic Hecke algebras associated with $G(r, 1, n)$. See [Ge1] and [J1], [J2]. On the other hand, control of actual modules is rather difficult in their approach, particularly in the cyclotomic case.[3] Hence, we have needed our approach in applications such as determination of representation type, and we are pursuing our direction further.[4]

2000 *Mathematics Subject Classification.* Primary 20C08, Secondary 05E10.

[1] By the Morita equivalence theorem for cyclotomic Hecke algebras proven by Dipper and Mathas [DM, Theorem 4.7], we may assume that $v_i \in q^{\mathbb{Z}}$, for $1 \le i \le r$.

[2] For the approach in [Gr], see [A3].

[3] Recently Geck has proved that finite Hecke algebras are cellular [Ge2]. Hence, they have better control of actual modules than before for finite Hecke algebras.

[4] We hope that a better understanding of the two approaches will lead to the merging of both theories.

Let $e$ be the multiplicative order of $q \neq 1$, $\mathfrak{g}$ the Kac-Moody Lie algebra of type $A^{(1)}_{e-1}$, $\{\Lambda_i \mid i \in \mathbb{Z}/e\mathbb{Z}\}$ the fundamental weights. We realize the Kashiwara crystal $B(\Lambda_i)$ on the set of $e$-restricted partitions. Suppose that $v_i = q^{\gamma_i}$, for $1 \leq i \leq r$. Then, our classification theorem asserts that simple modules are parametrized by the subset

$$B(\Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_r}) \subset B(\Lambda_{\gamma_1}) \otimes \cdots \otimes B(\Lambda_{\gamma_r}).$$

In particular, if $-Q = q^m$ then simple $\mathcal{H}_n(Q,q)$-modules are parametrized by $B(\Lambda_0 + \Lambda_m) \subset B(\Lambda_0) \otimes B(\Lambda_m)$. Further, when $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m)$, we can construct the corresponding simple module $D^{(\mu,\lambda)}$ as follows. Let $S^{(\mu,\lambda)}$ be the Specht module for $\mathcal{H}_n(Q,q)$ constructed by Dipper, James and Murphy in [DJM]. $S^{(\mu,\lambda)}$ is equipped with an invariant symmetric bilinear form. Then $D^{(\mu,\lambda)}$ is the module obtained from $S^{(\mu,\lambda)}$ by factoring out the radical of the bilinear form.

A bipartition $(\mu,\lambda)$ is called **Kleshchev** if $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m)$. The set of Kleshchev bipartitions may be computed by applying Kashiwara operators to the empty bipartition, but this does not give us an effective method of determining whether a given bipartition is Kleshchev or not.

The first purpose of this article is to give a non-recursive characterization of Kleshchev bipartitions. Our result is that $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m)$ if and only if $\mathrm{roof}(\mu) \subset \tau_m(\mathrm{base}(\lambda))$, where roof, base and $\tau_m$ are explicit operations on abacus displays. The definition of roof and base requires repeated application of up and down operations respectively, but roof and base are easily computable from a given partition.[5]

The characterization of $B(\Lambda_0 + \Lambda_m)$ as a subset of $B(\Lambda_0) \otimes B(\Lambda_m)$ is a purely crystal theoretic question. Due to a result of Littelmann, this characterization can be expressed in terms of his path model. Our strategy is to interpret his result in terms of the combinatorics of partitions. In his result, the initial direction and the final direction of a Lakshmibai-Seshadri path play an important role, and the crucial step in proving our theorem is to find a Young diagrammatic interpretation of these directions. Fortunately, the interpretation of the initial direction was already given in [KLMW1]. Here, we give the interpretation of the final direction. This suffices for proving our result for $m = 0$. Combined with arguments which interpret Littelmann's condition for different dominant integral weights, we reach our theorem.[6]

The second purpose of this article is to describe the crystal $B^w(\Lambda_m)$ for $w \in W$ in the same way that, in [KLMW1], $B_y(\Lambda_m)$ is described for $y \in W$. The work is motivated by standard monomial theory [LS], [L4]. In the Grassmannian case, see [KL] for a self-contained presentation in the spirit of the classical work of Hodge and Pedoe [H], [HP], and [KLMW2] for discussion of a similar approach for the affine Grassmannian.

The initial and final directions of a Lakshmibai-Seshadri path are related to the two types of Demazure crystals $B_y(\Lambda)$ and $B^w(\Lambda)$, for an integral dominant weight $\Lambda$. We explain the relationship in detail in section 6. The result for the

[5] Using this result, the first author and Jacon have settled a conjecture in [DJM] affirmatively. See [AJ].

[6] In the path model, an $e$-restricted partition is given by a sequence of $e$-cores and rational numbers. We show that the Mullineux map in the modular representation theory of the symmetric group and the Hecke algebra of type $A$ is given by conjugation of the $e$-cores. See Proposition 5.21 and the accompanying remark.

initial direction is due to Littelmann, and the result for the final direction is due to Kashiwara and Sagaki, who proved the result independently. We think that this self-contained explanation of the results benefits those who have an interest in Littelmann's path model.

The project started when the first author learned the idea of using Littelmann's result and the existence of [KLMW1] from Mark Shimozono. We are grateful to him. We are also grateful to Kashiwara for his permission to include his proof of the above mentioned result in this paper. Finally, the first author thanks Naito and Sagaki for explaining to him basic facts about Littelmann's path model, and Mathas and Fayers for explaining to him their results for $e = 2$ and $e = 3$, which give a different characterization of Kleshchev bipartitions without using Littelmann's result. We discuss their results in the last section.

## 2. Preliminaries

We assume that the reader is familiar with the theory of Kashiwara crystals. The three books [HK], [Jo] and [K1] are standard references. Throughout the paper, we always consider $\mathfrak{g}(A^{(1)}_{e-1})$-crystals, for fixed $e \geq 2$.

Let $\{\Lambda_m \mid m \in \mathbb{Z}/e\mathbb{Z}\}$ be the set of fundamental weights. We denote by $B(\Lambda_m)$ the Kashiwara crystal associated with $\Lambda_m$. Recall that a **partition** $\lambda$ is a sequence of non-increasing integers

$$\lambda_0 \geq \lambda_1 \geq \cdots$$

which has only a finite number of nonzero elements. We denote $\lambda_0$ by $a(\lambda)$. When $\lambda_{\ell-1} > 0$ and $\lambda_\ell = 0$, we denote $\lambda = (\lambda_0, \ldots, \lambda_{\ell-1})$ and denote $\ell$ by $\ell(\lambda)$. A partition is called $e$**-restricted** if $0 \leq \lambda_i - \lambda_{i+1} < e$, for all $i$.

We shall recall the realization of $B(\Lambda_m)$ in terms of $e$-restricted partitions. Let $\lambda$ be a partition. We color the nodes of $\lambda$ with the $e$ colors $\mathbb{Z}/e\mathbb{Z}$ by the following rule: let $x(a, b)$ be the node located on the $a^{th}$ row and the $b^{th}$ column. Then $x(a, b)$ has color $m - a + b + e\mathbb{Z}$. The number $m - a + b$ is called the **content** of $x(a, b)$, and the color $m - a + b + e\mathbb{Z}$ is called the **residue** of $x(a, b)$. Let $\lambda \subset \mu$ be a pair of partitions such that the number of nodes differs by one. Suppose that the residue of the node $x = \mu \setminus \lambda$ is $i$. Then we call $x$ an **addable** $i$**-node** of $\lambda$ and a **removable** $i$**-node** of $\mu$.

Let $B$ be the set of $e$-restricted partitions. We color the nodes of $\lambda \in B$ as above, and define

$$wt(\lambda) = \Lambda_m - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} N_i(\lambda)\alpha_i$$

where $N_i(\lambda)$ is the number of $i$-nodes in $\lambda$. In order to define two operators $\tilde{f}_i$ and $\tilde{e}_i$ on $B \sqcup \{0\}$, we read addable $i$-nodes and removable $i$-nodes from the first row to the last row and record the result as a sequence of $A$'s and $R$'s. Then we apply an algorithm which we call $RA$**-deletion**. Choose any $R \cdots A$, where the middle $\cdots$ means the letters which have been already deleted, and change it to $\cdot \cdots \cdot$. We repeat this procedure as many times as possible. The final sequence is of the form

$$\cdots A \cdots A \cdots A \cdots R \cdots R \cdots R \cdots R \cdots$$

where $\cdots$ is a sequence of dots of length greater than or equal to 0. The final sequence is uniquely determined (see [A1, Lemma 11.2]). The nodes which appear in the final sequence are called **addable normal** $i$**-nodes** and **removable normal** $i$**-nodes**. We define $\tilde{f}_i\lambda$ to be the partition obtained from $\lambda$ by adding the node which corresponds to the rightmost $A$ in the final sequence. If there is no $A$ in the final sequence, we set $\tilde{f}_i\lambda = 0$. Similarly, we define $\tilde{e}_i\lambda$ to be the partition obtained from $\lambda$ by removing the node which corresponds to the leftmost $R$ in the final sequence, and 0 if no $R$ exists in the final sequence. Finally, we define $\tilde{f}_i 0 = 0$ and $\tilde{e}_i 0 = 0$. Define

$$\varphi_i(\lambda) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k\lambda \neq 0\}, \;\; \epsilon_i(\lambda) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k\lambda \neq 0\}.$$

In other words, $\varphi_i(\lambda)$ is the number of $A$'s in the final sequence, and $\epsilon_i(\lambda)$ is the number of $R$'s in the final sequence.

The set $B$ with the additional data $wt$, $\epsilon_i$, $\varphi_i$, $\tilde{e}_i$ and $\tilde{f}_i$ is a realization of the crystal $B(\Lambda_m)$. This result is due to Misra and Miwa. See [A1, Theorem 11.11]. We denote the empty partition in $B(\Lambda_m)$ by $\emptyset_m$.

It is convenient to work with the abacus display of $\lambda$. The set of **beta numbers of charge** $m$ associated with $\lambda$ is, by definition, the set $J$ of decreasing integers

$$j_0 > j_1 > j_2 > \cdots > j_k > \cdots$$

defined by $j_k = \lambda_k + m - k$, for $k \geq 0$. It has the property that $j_k = m - k$, for $k >> 0$. We consider an abacus with $e$ runners

$$\begin{array}{cccc} . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 1 & \dots & e-1 \\ e & e+1 & \cdots & 2e-1 \\ . & . & \cdots & . \\ . & . & \cdots & . \end{array}$$

and put beads on the numbers $\{j_k \mid k \geq 0\}$. This is the **abacus display** of charge $m$ associated with $\lambda$.

**Example 2.1.** *Let $e = 3$, $m = 0$, and $\lambda = (4, 2, 1)$.*

*To read $J$ from $\lambda$, we look at each row and find the content of the node which is adjacent to the right end of the row.*

$$\begin{array}{ccccc} \times & \times & \times & \times & 4 \\ \times & \times & 1 & & \\ \times & -1 & & & \\ -3 & & & & \\ -4 & & & & \\ . & & & & \\ . & & & & \end{array}$$

*Thus, $J = \{4, 1, -1, -3, -4, \dots\}$, and the abacus display of $\lambda$ is as follows.*

$$\begin{array}{ccc} . & . & . \\ -6 & -5 & -4 \\ -3 & & -1 \\ & 1 & \\ & 4 & \end{array}$$

We call $j \in J$ with $j + e\mathbb{Z} = i+1$ a removable $i$-integer, and $j \in J$ with $j + e\mathbb{Z} = i$ an addable $i$-integer. The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ in terms of $J$ are given by the same procedure as above. We change the sequence $j_0, j_1, \dots$ to a sequence of dots, $R$'s, and $A$'s, and apply the $RA$-deletion as many times as possible. Note that a removable or addable integer $j \in J$ may not correspond to a removable or addable node of $\lambda$. However, this happens precisely when $\lambda_k = \lambda_{k+1}$. In this case, the content of the node which is adjacent to the right end of row $k$ is a removable $i$-integer, and the content of the node which is adjacent to the right end of row $k+1$ is an addable $i$-integer. In $RA$-deletion, these two adjacent values are removed from the final sequence.

The following definition is given in [KLMW1].

**Definition 2.2.** *Let $\lambda \in B(\Lambda_m)$ and $J$ the corresponding set of beta numbers of charge $m$. Let $U(J)$ be the set of beads which we may slide up by one in their runners. In other words,*

$$U(J) = \{x \in J \mid x - e \notin J\}.$$

*If $U(J) = \emptyset$ then define $\mathrm{up}(\lambda) = \lambda$. Suppose $U(J) \neq \emptyset$. Then set $p = \max U(J)$ and consider*

$$V(J) = \{x > p \mid x \notin p + e\mathbb{Z}, x - e \in J, x \notin J\}.$$

*Set $q = \min V(J)$. Then we define $\mathrm{up}(J)$ to be the set $(J \setminus \{p\}) \cup \{q\}$. That is, we obtain $\mathrm{up}(J)$ by moving the bead $p$ to $q$. We denote the corresponding partition by $\mathrm{up}(\lambda)$.*

**Example 2.3.** *Let $e = 3$, $m = 2$ and $\lambda = (3, 2, 1)$. Then the abacus display of $\lambda$ is*

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ -3 & -2 & -1 \\ & 1 & \\ 3 & & 5 \end{array}$$

*Then $U(J) = \{3, 5\}$ and $p = 5$. Thus $V(J) = \{6\}$ and $q = 6$. Therefore, $\mathrm{up}(J)$ is given by*

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ -3 & -2 & -1 \\ & 1 & \\ 3 & & \\ 6 & & \end{array}$$

*Thus, $\mathrm{up}(\lambda) = (4, 2, 1)$.*

**Lemma 2.4.** *Let $\lambda \in B(\Lambda_m)$.*
(1) *$\lambda \subset \mathrm{up}(\lambda)$.*
(2) *$\mathrm{up}(\lambda)$ is $e$-restricted.*
(3) *$\ell(\mathrm{up}(\lambda)) = \ell(\lambda)$.*

*Proof.* (1) Let $j'_0 > j'_1 > \cdots$ be the beta numbers of charge $m$ associated with $\mathrm{up}(\lambda)$. We set $j_{-1} = \infty$. Then, there exists $s \geq -1$ such that $j_s > q \geq j_{s+1}$. $q \notin J$ implies that $q > j_{s+1}$. Since $q > p$, there also exists $t > s$ such that $j_t = p$. Then, for $t > s+1$,

$$\begin{cases} j'_k = j_k \quad (0 \le k \le s) \\ j'_{s+1} = q > j_{s+1} \\ j'_k = j_{k-1} > j_k \quad (s+1 < k < t) \\ j'_t = j_{t-1} > j_t = p \\ j'_k = j_k \quad (k \ge t+1) \end{cases}$$

If $t = s+1$, replace the middle three lines with $j'_{s+1} = q > j_t = p$. In any case, $j'_k \ge j_k$, for all $k$. This implies the result.

(2) We only have to check the effect of removing $p$. We want to show $j'_t - j'_{t+1} \le e$. Since $\lambda$ is $e$-restricted and $p - e \notin J$, there exists $x \in \{p-e+1, \dots, p-1\} \cap J$. Note that $j_{t+1}$ is the largest element of $J$ which is smaller than $j_t = p$. Thus we have $x \le j_{t+1} = j'_{t+1}$.

Suppose first that $x + e \notin J$. Then $q \le x + e$, which implies that

$$x \le j'_{t+1} < j'_t \le j'_{s+1} = q \le x + e.$$

Thus, $\mathrm{up}(\lambda)$ is $e$-restricted.

Suppose next that $x + e \in J$. Then $j_t = p < x + e$ implies $j'_t = j_{t-1} \le x + e$. Thus $x \le j'_{t+1} < j'_t \le x + e$ and $\mathrm{up}(\lambda)$ is $e$-restricted.

(3) Let $s \in \mathbb{Z}$ be such that $\mathbb{Z}_{\le s} \subset J$ and $s+1 \notin J$. Then $\ell(\lambda) = |\{x \in J \mid x > s\}|$. As $p > s$ and $p$ moves to $q > p$, we have $\mathbb{Z}_{\le s} \subset J'$ and $s+1 \notin J'$, which implies $\ell(\mathrm{up}(\lambda)) = |\{x \in J' \mid x > s\}|$, and $\ell(\mathrm{up}(\lambda)) = \ell(\lambda)$. □

We remark that we may deduce $\lambda \subset \mathrm{up}(\lambda)$ from $|J' \cap \mathbb{Z}_{\ge a}| \ge |J \cap \mathbb{Z}_{\ge a}|$, for all $a \in \mathbb{Z}$. In fact, if there existed $k \ge 0$ such that $j'_0 = j_0, \dots, j'_{k-1} = j_{k-1}$ and $j'_k < j_k$, then we would obtain $|J' \cap \mathbb{Z}_{\ge j_k}| < |J \cap \mathbb{Z}_{\ge j_k}|$, a contradiction.

If we apply the up operation successively, then we reach $U(J) = \emptyset$ after finitely many steps. To see this, choose $s$ such that $\mathbb{Z}_{\le s} \subset J$. Then $\mathbb{Z}_{\le s} \subset \mathrm{up}(J)$. Thus, $\mathbb{Z}_{\le s}$ remains untouched during the successive applications of up operations. Let $N$ be the number of elements in $\{x \in J \mid x > s\}$ and $K = \mathbb{Z}_{\le s} \cup \{s + ke \mid 1 \le k \le N\}$. We write $J \le J'$ if $j_k \le j'_k$, for all $k \ge 0$. Note that if $J$ is the set of beta numbers associated with an $e$-restricted partition and of the form $J = \mathbb{Z}_{\le s} \cup \{j_0, \dots, j_{N-1}\}$, where $j_0 > \dots > j_{N-1} > s$, then $J \le K$. Thus, we have $\mathrm{up}^i(J) \le K$, for all $i \ge 0$. As the sequence $J, \mathrm{up}(J), \mathrm{up}^2(J), \dots$ is strictly increasing as long as $U(J) \ne \emptyset$, we reach $U(J) = \emptyset$ after finitely many steps.

This allows us to define $\mathrm{roof}(J)$ as follows.

**Definition 2.5.** *Let $\lambda \in B(\Lambda_m)$ and let $J$ be as before. Apply the up operation to $J$ until $U(J) = \emptyset$. We denote the resulting* $\mathrm{up}^{max}(J)$ *by* $\mathrm{roof}(J)$*, and denote the corresponding partition by* $\mathrm{roof}(\lambda)$*.*

Note that by definition, $\mathrm{roof}(\lambda)$ is an $e$-core.

**Definition 2.6.** *Let $\lambda \in B(\Lambda_m)$ and $J$ the corresponding set of beta numbers of charge $m$. Let $U(J)$ be as before. If $U(J) = \emptyset$ then define* $\mathrm{down}(\lambda) = \lambda$*. Suppose $U(J) \ne \emptyset$. Then set $p' = \min U(J)$ and consider*

$$W(J) = \{x > p' - e \mid x \in J, x + e \notin J\} \cup \{p'\}.$$

*Set $q' = \min W(J)$. Then we define* $\mathrm{down}(J) = (J \setminus \{q'\}) \cup \{p' - e\}$*. That is, we obtain* $\mathrm{down}(J)$ *by moving the bead $q'$ to $p' - e$. We denote the corresponding partition by* $\mathrm{down}(\lambda)$*.*

**Lemma 2.7.** *Let $\lambda \in B(\Lambda_m)$.*
(1) $\lambda \supset \mathrm{down}(\lambda)$.
(2) $\mathrm{down}(\lambda)$ *is $e$-restricted.*
(3) $a(\mathrm{down}(\lambda)) = a(\lambda)$.

*Proof.* (1) Let $j'_0 > j'_1 > \cdots$ be the beta numbers of charge $m$ associated with $\mathrm{down}(\lambda)$. Then, there exists $s \geq 0$ such that $j_s > p' - e > j_{s+1}$, and there exists $0 \leq t \leq s$ such that $j_t = q'$. Now,

$$\begin{cases} j'_k = j_k \quad (0 \leq k < t) \\ j'_t = j_{t+1} < j_t = q' \\ j'_k = j_{k+1} < j_k \quad (t < k < s) \\ j'_s = p' - e < j_s \\ j'_k = j_k \quad (k \geq s+1) \end{cases}$$

We replace the middle three lines with $j'_t = p' - e < j_t = q'$ when $t = s$. Thus $j'_k \leq j_k$, for all $k$. This implies the result.

(2) We only have to consider the effect of removing $q'$ as before. We want to show $j'_{t-1} - j'_t \leq e$. Note that there exists $x \in \{p'-e+1, \ldots, p'-1\} \cap J$ since $\lambda$ is $e$-restricted and $p' - e \notin J$.

Suppose first that $q' \neq p'$. Then $p' \geq j'_{t-1}$ since $p' > q'$ and $j'_{t-1} = j_{t-1}$ is the smallest element of $J$ which is greater than $j_t = q'$. Thus

$$p' - e = j'_s \leq j'_t < j'_{t-1} \leq p'.$$

Suppose next that $q' = p'$. There exists $x \in \{p'-e+1, \ldots, p'-1\} \cap J$ as before. As $x < p' = j_t$ and $x \in J$, we have $x \leq j_{t+1}$. On the other hand, $q' = p'$ implies that $x + e \notin J$ is impossible. Thus, $j_t = p' < x + e$ implies $j_{t-1} \leq x + e$ and

$$x \leq j_{t+1} = j'_t < j'_{t-1} = j_{t-1} \leq x + e.$$

(3) As $a(\lambda) = j_0 - m$ and $a(\mathrm{down}(\lambda)) = j'_0 - m$, we show $j_0 = j'_0$. If $p' < j_0$ then $q' \leq p' < j_0$. If $p' = j_0$ then $j_0 - e \notin J$ and, since $\lambda$ is $e$-restricted, there exists $x \in J$ such that $j_0 - e < x < j_0$. Then, as $x + e \notin J$, $x \in W(J)$ and $q' \leq x < j_0$. Hence $q' < j_0$ in both cases and $q'$ moves to $p' - e < q'$. Thus $j_0 = j'_0$. $\square$

As before, we may deduce $\lambda \supset \mathrm{down}(\lambda)$ from $|J' \cap \mathbb{Z}_{\geq a}| \leq |J \cap \mathbb{Z}_{\geq a}|$ for all $a \in \mathbb{Z}$.

We apply the down operation successively. It is easy to see that we reach $U(J) = \emptyset$ after finitely many steps: the size of the corresponding partition strictly decreases as long as $U(J) \neq \emptyset$. In section 7, we need a better understanding of how the value $p'$ changes during the process. Thus, we analyze it in detail here.

Suppose that we apply the down operation to $J^{old}$ to obtain $J^{new}$ and that $U(J^{old}) \neq \emptyset$ and $U(J^{new}) \neq \emptyset$. Since ${p'}^{old} \notin U(J^{new})$ implies ${p'}^{new} \neq {p'}^{old}$, we have either ${p'}^{new} > {p'}^{old}$ or ${p'}^{new} < {p'}^{old}$.

Suppose that ${p'}^{new} < {p'}^{old}$. If ${p'}^{new} - e \notin J^{old}$ then ${p'}^{new} \notin J^{old}$ as ${p'}^{new} \in J^{old}$ would imply ${p'}^{new} \geq {p'}^{old}$. Hence ${p'}^{new} \in J^{new} \setminus J^{old}$ and we have ${p'}^{new} = {p'}^{old} - e$.

The set $U(J)$ changes in the following way. Let $q' = \min W(J^{old})$.

(a) If $q' < {p'}^{old}$ then $q' - ke \in J^{old}$, for all $k \geq 0$, and $q' + e \notin J^{old}$. Hence,

$$U(J^{new}) \setminus \{{p'}^{new}\} \subset U(J^{old}) \setminus \{{p'}^{old}\}.$$

(b) If $q' = {p'}^{old}$ then

$$U(J^{new}) \setminus \{{p'}^{new}\} \subset \Big(U(J^{old}) \setminus \{{p'}^{old}\}\Big) \cup \{{p'}^{old} + e\}.$$

Next suppose that $p'$ starts decreasing at $p_0 = \min U(J_0)$ and stops decreasing at $p_N = \min U(J_N)$. By the above consideration, the innovation of $p'$ is given by the recursion ${p'}^{new} = {p'}^{old} - e$, so $p_k = p_0 - ke$, for $0 \le k \le N$. Denote $q_k = \min W(J_k)$. Define $s \ge 0$ by $q_k = p_k$, for $0 \le k < s$, and $q_s \ne p_s$. We shall show by induction on $k$ that

$$U(J_k) \cap \mathbb{Z}_{\le p_0} = \{p_0 - ke\}, \quad \text{for } 0 \le k \le N.$$

For $0 \le k < s$, $J_{k+1}$ is obtained from $J_k$ by sliding the bead at $p_0 - ke$ up to $p_0 - (k+1)e$. Thus, if $k \ge 1$ and $x \le p_0$ is such that $x \in J_{k+1}$ and $x + e\mathbb{Z} = p_0 + e\mathbb{Z}$, then $x \le p_0 - (k+1)e$. Suppose that ${p'}^{old} + e \in U(J^{new})$ occured at $k \ge 1$. Thus, ${p'}^{new} = p_0 - (k+1)e$ and ${p'}^{old} + e = p_0 - (k-1)e$. Let $x = {p'}^{old} + e$. Then $x \le p_0$ satisfies $x \in J_{k+1}$ and $x + e\mathbb{Z} = p_0 + e\mathbb{Z}$ but $x > p_0 - (k+1)e$. Thus, ${p'}^{old} + e \notin U(J^{new})$ and

$$U(J_{k+1}) \setminus \{p_{k+1}\} \subset U(J_k) \setminus \{p_k\} \subset \mathbb{Z}_{\ge p_0 + 1}$$

by the induction hypothesis.

For $s \le k \le N$, we have $q_k < p_k$. If $k = s$ then this is by definition. Suppose that $q_k < p_k$. Then $q_k - e \in J_k$ and $q_k \notin J_{k+1}$, $p_{k+1} - e = p_k - 2e < q_k - e$ imply $q_k - e \in W(J_{k+1})$. Hence,

$$q_{k+1} \le q_k - e < p_k - e = p_{k+1}.$$

Therefore, we have

$$U(J_{k+1}) \setminus \{p_{k+1}\} \subset U(J_k) \setminus \{p_k\} \subset \mathbb{Z}_{\ge p_0 + 1},$$

for $s \le k < N$. We have proved that $U(J_k) \cap \mathbb{Z}_{\le p_0} = \{p_k\}$, for $0 \le k \le N$.

Now, set $J^{old} = J_N$ and ${p'}^{old} = p_N$. Then we obtain $J^{new}$ from $J^{old}$ by moving $q_N$ to $p_N - e$. Suppose that $U(J^{new}) \ne \emptyset$. Then ${p'}^{new} > {p'}^{old}$.

We claim that ${p'}^{new} > p_0$. In fact, as $p_N - e \notin U(J^{new})$, we have either ${p'}^{new} \in U(J_N) \setminus \{p_N\}$ or ${p'}^{new} = q_N + e$. In the former case, $U(J_N) \cap \mathbb{Z}_{\le p_0} = \{p_N\}$ implies ${p'}^{new} > p_0$. Suppose that ${p'}^{new} = q_N + e \le p_0$. If $q_k < p_k$ for some $k \le N$, then $q_N < p_N$, and $p_N < q_N + e \notin J_N$ implies $q_N + e \notin J^{new}$, which contradicts ${p'}^{new} \in J^{new}$. If $q_k = p_k$ for $0 \le k \le N$, then $p_0 - ke$ is not contained in $J^{new}$, for $0 \le k \le N$. So $q_N + e = p_0 - (N-1)e \notin J^{new}$ either. We have proved that ${p'}^{new} > p_0$.

As we reach $U(J) = \emptyset$ after finitely many steps, we may define $\mathrm{base}(J)$ as follows.

**Definition 2.8.** *Let $\lambda \in B(\Lambda_m)$ and let $J$ be as before. Apply the down operation to $J$ until $U(J) = \emptyset$. We denote the resulting* $\mathrm{down}^{max}(J)$ *by* $\mathrm{base}(J)$*, and denote the corresponding partition by* $\mathrm{base}(\lambda)$.

Note that $\mathrm{base}(\lambda)$ is an $e$-core by definition.

## 3. Weyl group action

Let $B$ be a $\mathfrak{g}$-crystal and $W$ the corresponding Weyl group. In our case of $B(\Lambda_m)$, $W$ is the Coxeter group generated by $\{s_i \mid i \in \mathbb{Z}/e\mathbb{Z}\}$ subject to $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ otherwise.

**Theorem 3.1** ([K1, Theorem 9.4.1]). *Let $B$ be a normal crystal. Then the following defines a $W$-action on $B$.*

$$s_i b = \begin{cases} \tilde{f}_i^{wt(b)(h_i)} b & (if\ \ wt(b)(h_i) \geq 0) \\ \tilde{e}_i^{-wt(b)(h_i)} b & (if\ \ wt(b)(h_i) \leq 0) \end{cases}$$

*Further, $wt(s_i b) = s_i(wt(b)) = wt(b) - wt(b)(h_i)\alpha_i$.*

Recall that $B(\Lambda_m)$ is a normal crystal. Hence, we have a $W$-action and

$$\tilde{e}_i^{max}\lambda = \tilde{e}_i^{\epsilon_i(\lambda)}\lambda, \quad \tilde{f}_i^{max}\lambda = \tilde{f}_i^{\varphi_i(\lambda)}\lambda.$$

**Definition 3.2.** *Let $\lambda \in B(\Lambda_m)$. We say that $\lambda$ is an $s_i$-**core** if $x \in U(J)$ implies $x + e\mathbb{Z} \neq i$ and $x + e\mathbb{Z} \neq i + 1$.*

Thus, $\lambda$ is an $e$-core if and only if it is an $s_i$-core, for all $i \in \mathbb{Z}/e\mathbb{Z}$.

**Lemma 3.3.** *Suppose that $\lambda \in B(\Lambda_m)$.*

(1) *Let $A_i(\lambda)$ and $R_i(\lambda)$ be the set of addable $i$-nodes and the set of removable $i$-nodes of $\lambda$ respectively. Then*

$$wt(\lambda)(h_i) = |A_i(\lambda)| - |R_i(\lambda)|.$$

(2) *Assume that $\lambda$ is an $s_i$-core. Then either*
  - (i) *$A_i(\lambda) = \emptyset$ and $s_i\lambda = \tilde{e}_i^{max}\lambda = \lambda \setminus \{$all removable $i$-nodes$\}$, or*
  - (ii) *$R_i(\lambda) = \emptyset$ and $s_i\lambda = \tilde{f}_i^{max}\lambda = \lambda \cup \{$all addable $i$-nodes$\}$.*

*Proof.* (1) is proved by induction on $|\lambda|$. If $\lambda = \emptyset_m$, $\Lambda_m(h_i) = \delta_{im}$ proves the result. Suppose that $\lambda = \mu \cup \{x\}$ and the residue of $x$ is $j$. Thus $wt(\lambda) = wt(\mu) - \alpha_j$. Note that

$$wt(\lambda)(h_i) = \begin{cases} wt(\mu)(h_i) & (j \neq i, i \pm 1) \\ wt(\mu)(h_i) + 1 & (j = i \pm 1) \\ wt(\mu)(h_i) - 2 & (j = i) \end{cases}$$

Checking how $A_i(\mu)$ and $R_i(\mu)$ change when $x$ is added, we obtain the result.

(2) For a hook $\Gamma = (a, 1^r)$, the $a$ nodes consist the arm of $\Gamma$ and the $r$ nodes consist the leg of $\Gamma$. The residue of the lowest node of the leg is called the residue of $\Gamma$. Let $J$ be the set of beta numbers of charge $m$ associated with $\lambda$. Recall that sliding a bead in $J$ on the $i^{th}$ runner up by one is the same as removing an $e$-hook $\Gamma$ whose residue is $i$. Suppose that there exist $x \in A_i(\lambda)$ and $y \in R_i(\lambda)$ such that $x$ is in the $j^{th}$ row of $\lambda$ and $y$ is in the $k^{th}$ row of $\lambda$. If $j < k$ then we may remove at least one $e$-hook of residue $i$ from $\lambda$. Similarly, if $j > k$ then we may remove at least one $e$-hook of residue $i + 1$ from $\lambda$. Since $\lambda$ is an $s_i$-core, both cannot occur. In other words, one of $A_i(\lambda)$ or $R_i(\lambda)$ must be empty. Thus, $RA$-deletion does not occur, which implies that either $\epsilon_i(\lambda) = |R_i(\lambda)|$ and $\varphi_i(\lambda) = 0$, or $\epsilon_i(\lambda) = 0$ and $\varphi_i(\lambda) = |A_i(\lambda)|$ respectively. Now the result follows from (1). □

We show that this Weyl group action coincides that of [KLMW1] on $e$-cores.

**Lemma 3.4.** *Let $\lambda$ be an $s_i$-core, $J$ the corresponding set of beta numbers of charge $m$. We denote by $s_iJ$ the set of beta numbers of charge $m$ associated with $s_i\lambda$.*

(1) *If $i \neq 0$ then $s_iJ$ is obtained by switching the $i^{th}$ and $(i + 1)^{th}$ runners.*

(2) *The $(e - 1)^{th}$ runner of $s_0J$ is obtained from the $0^{th}$ runner of $J$ by sliding up by one. Similarly, the $0^{th}$ runner of $s_0J$ is obtained from the $(e - 1)^{th}$ runner of $J$ by sliding down by one.*

(3) *$s_i\lambda$ is an $s_i$-core.*

*Proof.* (1) If the length of the $i^{th}$ runner of $J$ exceeds that of $(i+1)^{th}$ runner by $k$, these $k$ beads correspond to addable $i$-nodes of $\lambda$. Thus, Lemma 3.3 (2) implies that $s_i\lambda$ is obtained from $\lambda$ by adding all the addable $i$-nodes. The resulting $s_iJ$ is the same as the one which is obtained by switching the two runners. If the length of the $(i+1)^{th}$ runner of $J$ exceeds that of $i^{th}$ runner by $k$, these $k$ beads correspond to removable $i$-nodes of $\lambda$. Thus, Lemma 3.3 (2) again implies that $s_iJ$ is obtained from $J$ by switching the two runners.

The proof of (2) is entirely similar to that of (1) and (3) is an obvious consequence of (1) and (2). □

The following proposition seems to be well-known, but we could not find a reference.

**Proposition 3.5.** *The set of $e$-cores in $B(\Lambda_m)$ coincides the $W$-orbit through $\emptyset_m$.*

*Proof.* We can prove that an $e$-core belongs to $W\emptyset_m$ by induction on $|\lambda|$. Let $x$ be the right end of the last row of $\lambda$, and let $i$ be the residue of $x$. Set $\mu = \tilde{e}_i^{max}\lambda$. Then $|\mu| < |\lambda|$ since $x$ is a removable normal $i$-node, and $\lambda = \tilde{f}_i^{max}\mu$ since $\lambda$ is an $e$-core. Since the set of $e$-cores is stable under $W$-action by Lemma 3.4 (3), $\mu$ is again an $e$-core, so $\mu \in W\emptyset_m$ by the induction hypothesis. Thus, we have $\lambda = s_i\mu \in W\emptyset_m$. Since a non-empty $W$-stable subset of a $W$-orbit must coincide with the $W$-orbit itself, we have the result. □

**Definition 3.6.** *Let $W_m$ be the subgroup of $W$ generated by $\{s_i \mid i \neq m\}$. We denote by $W/W_m$ the set of distinguished coset representatives.*

As $W_m$ is the Coxeter group of type $A_{e-1}$, $W_m$ has the longest element. Thus the following definition makes sense.

**Definition 3.7.** *We denote by $w_m$ the longest element of $W_m$.*

Recall that $W$ becomes a poset by the Bruhat-Chevalley order. We write $u \le v$, for $u, v \in W$. By virtue of Proposition 3.5, each $e$-core $\lambda \in B(\Lambda_m)$ can be written in the form $\lambda = w\emptyset_m$, for $w \in W/W_m$, in a unique manner.

## 4. Demazure crystal

Following [K1] and [K3], we introduce two types of Demazure crystals.

**Definition 4.1.** *Let $y, w \in W$ and let $y = s_{i_1}\cdots s_{i_\ell}$ be a reduced expression for $y$. Then we define $B_y(\Lambda_m)$ and $B^w(\Lambda_m)$ as follows.*

$$B_y(\Lambda_m) = \{\tilde{f}_{i_1}^{a_1}\cdots\tilde{f}_{i_\ell}^{a_\ell}\emptyset_m \mid (a_1,\dots,a_\ell) \in (\mathbb{Z}_{\ge 0})^\ell\} \setminus \{0\},$$
$$B^w(\Lambda_m) = \{b \in B(\Lambda_m) \mid G_v(b) \in U_v^-(\mathfrak{g})u_{w\Lambda_m}\}.$$

By [K1, Proposition 9.1.3, 9.1.5], $B_y(\Lambda_m)$ does not depend on the choice of the reduced expression. For the notations $G_v(b)$ and $u_{w\Lambda_m}$, see §6.

The following are fundamental properties of the Demazure crystals. The results hold for any dominant integral weight.

**Proposition 4.2** ([K3, Proposition 3.2.3, 3.2.4, 4.3, 4.4])**.**
(1) *$\tilde{e}_iB_y(\Lambda_m) \subset B_y(\Lambda_m) \cup \{0\}$ and $\tilde{f}_iB^w(\Lambda_m) \subset B^w(\Lambda_m) \cup \{0\}$.*
(2) *If $s_iy < y$ then $B_y(\Lambda_m) = \cup_{k\ge 0}\tilde{f}_i^kB_{s_iy}(\Lambda_m) \setminus \{0\}$.*

(3) *If* $s_iw > w$ *then* $B^w(\Lambda_m) = \cup_{k\ge 0}\tilde{e}_i^k B^{s_iw}(\Lambda_m)\setminus\{0\}$.
(4) *Let* $y, w \in W/W_m$. *Then the following are equivalent.*
   (i) $y \ge w$.
   (ii) $B^w(\Lambda_m)\cap B_y(\Lambda_m) \ne \emptyset$.
   (iii) $B_w(\Lambda_m) \subset B_y(\Lambda_m)$.
   (iv) $w\emptyset_m \in B_y(\Lambda_m)$.
   (v) $B^y(\Lambda_m) \subset B^w(\Lambda_m)$.
   (vi) $y\emptyset_m \in B^w(\Lambda_m)$.

Next theorem is the main result of [KLMW1]. However, the proof we will give is slightly different from the original: see Theorem 6.2, Theorem 6.3 and Corollary 6.4.

**Theorem 4.3** ([KLMW1, Theorem 1.1])**.** *In the partition realization of* $B(\Lambda_m)$*, we have*

$$B_y(\Lambda_m) = \{\lambda \in B(\Lambda_m) \mid \mathrm{roof}(\lambda) \subset y\emptyset_m\}.$$

**Proposition 4.4.** *Let* $\lambda = u\emptyset_m$ *and* $\mu = v\emptyset_m$*, for* $u, v \in W$.
(1) *If* $u \le v$ *then* $\lambda \subset \mu$.
(2) *If* $\lambda \subset \mu$ *and* $u, v \in W/W_m$ *then* $u \le v$.

*Proof.* (1) We prove this by induction on $\ell(v)$. Let $v = s_i s_{i_2}\cdots s_{i_\ell}$ be a reduced expression. Then $u$ is a subword of the expression.

First we suppose that the leftmost $s_i$ does not appear in this subword. Then $u \le s_iv$ and the induction hypothesis implies that

$$\lambda = u\emptyset_m \subset s_iv\emptyset_m = s_i\mu.$$

Write $w = s_iv$. Then $w < s_iw$ since $s_iv < v$. If $w^{-1}\alpha_i$ were a negative root, then the standard argument would show that $w > s_iw$. Hence $w^{-1}\alpha_i$ is a positive root. In other words, $v^{-1}\alpha_i$ is a negative root and $\langle\Lambda_m, v^{-1}h_i\rangle \le 0$. We have

$$wt(s_i\mu) = wt(s_iv\emptyset_m) = s_iv\Lambda_m = v\Lambda_m - \langle\Lambda_m, v^{-1}h_i\rangle\alpha_i.$$

Hence $wt(s_i\mu) - wt(\mu) \in \sum_{j\in\mathbb{Z}/e\mathbb{Z}}\mathbb{Z}_{\ge 0}\alpha_j$. Note that

$$\begin{cases} wt(\mu) &= \Lambda_m - \sum_{j\in\mathbb{Z}/e\mathbb{Z}} N_j(\mu)\alpha_j, \\ wt(s_i\mu) &= \Lambda_m - \sum_{j\in\mathbb{Z}/e\mathbb{Z}} N_j(s_i\mu)\alpha_j. \end{cases}$$

Thus $|s_i\mu| \le |\mu|$. In particular, $\mu$ is obtained from $s_i\mu$ by adding all addable $i$-nodes by Lemma 3.3 (2). Hence $\lambda \subset s_i\mu \subset \mu$.

Next suppose that the leftmost $s_i$ appears in the subword for $u$. Then $s_iu \le s_iv$ and the induction hypothesis implies $s_i\lambda \subset s_i\mu$. Note that $s_iu < u$ and $s_iv < v$. Thus, the same argument as above shows that $\lambda$ and $\mu$ are obtained from $s_i\lambda$ and $s_i\mu$ by adding all addable $i$-nodes, respectively. If an addable $i$-node of $s_i\lambda$ is contained in $s_i\mu$, it is contained in $s_i\mu$ and hence in $\mu$. If an addable $i$-node of $s_i\lambda$ is not contained in $s_i\mu$, then it is also an addable $i$-node of $s_i\mu$. Thus, it is contained in $\mu$. We have proved $\lambda \subset \mu$.

(2) We prove this by induction on $\ell(v)$ as above. If $\lambda = \mu$ then there is nothing to prove. So assume that $\lambda \ne \mu$. Pick a removable node of the skew shape $\mu/\lambda$ and denote its residue by $i$. As $\mu$ is an $e$-core, $s_i\mu \subset \mu$ and $s_i\mu \ne \mu$. Thus we have $s_iv < v$ by (1).

We show that we have either $\lambda \subset s_i\mu$ or $s_i\lambda \subset s_i\mu$. Suppose that $\lambda \not\subset s_i\mu$. Then any node $x \in \lambda \setminus s_i\mu \subset \mu \setminus s_i\mu$ is a removable $i$-node of $\lambda$. Thus $s_i\lambda \subset s_i\mu$ follows. Hence, we consider these two cases.

First suppose that $\lambda \subset s_i\mu$. Then the induction hypothesis implies that $u \le s_i v$, as $u$ is a distinguished coset representative. Thus $u \le v$.

Next suppose that $s_i\lambda \subset s_i\mu$ and $\lambda \not\subset s_i\mu$. Then $s_i\lambda \supset \lambda$ does not occur. Hence, $s_i\lambda \subset \lambda$ and $s_i\lambda \ne \lambda$, which implies $s_i u < u$ as before.

Write $s_i u = u' t$, where $u' \in W/W_m$ and $t \in W_m$. Let $u' = s_{i_1} \cdots s_{i_p}$ and $t = s_{j_1} \cdots s_{j_q}$ be reduced expressions of $u'$ and $t$ respectively. Then, as

$$u = s_i s_{i_1} \cdots s_{i_p} s_{j_1} \cdots s_{j_q} \text{ and } \ell(u) = \ell(s_i u) + 1 = \ell(u') + \ell(t) + 1 = p + q + 1,$$

this is a reduced expression of $u$. Since $u$ is a distinguished coset representative, we have $q = 0$ and $s_i u$ is distinguished. Now the induction hypothesis implies $s_i u \le s_i v$. As $s_i(s_i v) > s_i v$, we have $u \le v$ as desired. □

**Corollary 4.5.** *Write* $\mathrm{roof}(\lambda) = y_\lambda \emptyset_m$, *for a unique* $y_\lambda \in W/W_m$. *Then*

$$y_\lambda = \min \{ y \in W \mid \lambda \in B_y(\Lambda_m) \}$$

*with respect to the Bruhat-Chevalley order.*

*Proof.* If $\lambda \in B_y(\Lambda_m)$ then Theorem 4.3 shows that $\mathrm{roof}(\lambda) \subset y\emptyset_m$. Then Proposition 4.4 implies that $y_\lambda \le y$. As $\mathrm{roof}(\lambda) \subset y_\lambda \emptyset_m$, we have $\lambda \in B_{y_\lambda}(\Lambda_m)$ and $y_\lambda$ is the unique minimal element of $\{ y \in W \mid \lambda \in B_y(\Lambda_m) \}$. □

## 5. Littelmann's path model

Littelmann introduced a realization of Kashiwara crystals in terms of $W$. [NS2, §1] is a concise review of the path model. The results of this section hold for a general dominant integral weight, but we state them only for $\Lambda_m$.

**Definition 5.1.** *Let* $\mu \ne \nu \in W\Lambda_m$ *be two weights. If there exists a sequence of positive real roots* $\beta_1, \ldots, \beta_r$ *such that*

$$\langle s_{\beta_{j-1}} \cdots s_{\beta_1} \mu, h_{\beta_j} \rangle \in \mathbb{Z}_{<0},$$

*for* $1 \le j \le r$ *and* $\nu = s_{\beta_r} s_{\beta_{r-1}} \cdots s_{\beta_1} \mu$, *then we write* $\mu > \nu$. *Here,* $h_{\beta_j}$ *is the coroot of* $\beta_j$.

*Let* $0 < a < 1$ *be a rational number. A sequence*

$$\mu,\ s_{\beta_1}\mu,\ s_{\beta_2}s_{\beta_1}\mu,\ \cdots,\ s_{\beta_r}s_{\beta_{r-1}} \cdots s_{\beta_1}\mu = \nu$$

*with* $r$ *maximal is called an* $a$-**chain** *if*

$$\langle s_{\beta_{j-1}} \cdots s_{\beta_1} \mu, h_{\beta_j} \rangle \in a^{-1}\mathbb{Z}_{<0},$$

*for all* $j$.

If $\mu = y\Lambda_m$ and $\nu = w\Lambda_m$ for $y, w \in W/W_m$, then $\mu > \nu$ is equivalent to $y > w$.

**Lemma 5.2** ([L2, Lemma 4.1])**.**
(1) *If* $\mu \ge \nu$ *is such that* $\mu(h_i) < 0$ *and* $\nu(h_i) \ge 0$, *then* $s_i\mu \ge \nu$.
(2) *If* $\mu \ge \nu$ *is such that* $\mu(h_i) \le 0$ *and* $\nu(h_i) > 0$, *then* $\mu \ge s_i\nu$.

Let $0 = a_0 < a_1 < \cdots < a_s = 1$ and $\nu_1, \ldots, \nu_s \in W\Lambda_m$. We consider a piecewise linear path $\pi(t)$, for $0 \le t \le 1$, which takes values in the dual space of the Cartan subalgebra:

$$\pi(t)|_{[a_{j-1},a_j]} = \sum_{k=1}^{j-1} (a_k - a_{k-1})\nu_k + (t - a_{j-1})\nu_j.$$

In other words, we start with the origin, and change direction from $\nu_j$ to $\nu_{j+1}$ at $t = a_j$, for $1 \le j < s$.

**Definition 5.3.** *The piecewise linear path $\pi(t)$ given by $(\nu_1, \ldots, \nu_s; a_0, \ldots, a_s)$ as above, is a* **Lakshmibai-Seshadri path***, if the following hold for all $j$.*

(i) *$a_j$ is a rational number and $\nu_j > \nu_{j+1}$.*
(ii) *There exists an $a_j$-chain for $\nu_j > \nu_{j+1}$.*

*We denote the set of Lakshmibai-Seshadri paths by $\mathbb{B}(\Lambda_m)$.*

We call Lakshmibai-Seshadri paths **LS paths** for short.

**Definition 5.4.** *Let $\pi \in \mathbb{B}(\Lambda_m)$ be given by $(\nu_1, \ldots, \nu_s; a_0, \ldots, a_s)$. We call $\nu_1$ the* **initial direction** *of $\pi$ and denote it by $i(\pi)$. Similarly, we call $\nu_s$ the* **final direction** *and denote it by $f(\pi)$.*

**Definition 5.5.** *We say that $\pi(t)$ satisfies the* **integrality condition** *if the minimum value of $\pi(t)(h_i)$ is an integer, for all $i$.*

**Lemma 5.6** ([L2, Lemma 4.5(d)])**.** *The LS-paths satisfy the integrality condition.*

Define $Q = \min\{\pi(t)(h_i) \mid 0 \le t \le 1\}$. We shall define the operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathbb{B}(\Lambda_m) \sqcup \{0\}$. First of all, we set $\tilde{e}_i\pi = 0$ if $Q > -1$, and $\tilde{f}_i\pi = 0$, if $Q > \pi(1)(h_i) - 1$.

Suppose that $Q \le -1$. Then define

$$t_1 = \min\{t \in [0,1] \mid \pi(t)(h_i) = Q\}$$
$$t_0 = \max\{t \in [0,t_1] \mid \pi(t)(h_i)|_{[0,t]} \ge Q + 1\}$$

and reflect the path $\pi(t)$ for the interval $[t_0, t_1]$ to define:

$$(\tilde{e}_i\pi)(t) = \begin{cases} \pi(t) & (0 \le t \le t_0) \\ s_i(\pi(t) - \pi(t_0)) + \pi(t_0) & (t_0 \le t \le t_1) \\ \pi(t) + \alpha_i & (t_1 \le t \le 1) \end{cases}$$

Suppose that $Q \le \pi(1)(h_i) - 1$. Then define

$$t_0 = \max\{t \in [0,1] \mid \pi(t)(h_i) = Q\}$$
$$t_1 = \min\{t \in [t_0,1] \mid \pi(t)(h_i)|_{[t,1]} \ge Q + 1\}$$

and define:

$$(\tilde{f}_i\pi)(t) = \begin{cases} \pi(t) & (0 \le t \le t_0) \\ s_i(\pi(t) - \pi(t_0)) + \pi(t_0) & (t_0 \le t \le t_1) \\ \pi(t) - \alpha_i & (t_1 \le t \le 1) \end{cases}$$

We then define $wt(\pi) = \pi(1)$ and

$$\epsilon_i(\pi) = -Q, \quad \varphi_i(\pi) = \pi(1)(h_i) - Q.$$

Then, by [Jo, Corollary 6.4.27] or [K5, Theorem 4.1], the set $\mathbb{B}(\Lambda_m)$ with the additional data $wt$, $\epsilon_i$, $\varphi_i$, $\tilde{e}_i$ and $\tilde{f}_i$ is a realization of the crystal $B(\Lambda_m)$. The isomorphism of the two realizations, one by $e$-restricted partitions, the other by the LS-paths, is unique. Thus, we identify the two realizations and sometimes write $\lambda = (\nu_1, \dots, \nu_s; a_0, \dots, a_s)$, for an $e$-restricted partition $\lambda$. We denote $\nu_1$ and $\nu_s$ by $i(\lambda)$ and $f(\lambda)$ respectively.

The following is one of the key results we use in this paper.

**Theorem 5.7** ([L3, Theorem 10.1])**.** *Let*

$$\pi = \pi^{(1)} \otimes \cdots \otimes \pi^{(r)} \in B(\Lambda_{m_1}) \otimes \cdots B(\Lambda_{m_r}).$$

*Then $\pi$ belongs to $B(\Lambda_{m_1} + \cdots \Lambda_{m_r})$ if and only if there exists a sequence*

$$w_1^{(1)} \geq \cdots \geq w_{N_1}^{(1)} \geq w_1^{(2)} \geq \cdots \geq w_{N_2}^{(2)} \geq \cdots\cdots \geq w_{N_r}^{(r)}$$

*in $W$ such that*

$$\pi^{(k)} = (w_1^{(k)}\Lambda_{m_k}, \dots, w_{N_k}^{(k)}\Lambda_{m_k}; a_0^{(k)}, \dots, a_{N_k}^{(k)}),$$

*for $1 \leq k \leq r$.*

Recall that $w_0$ is the longest element of $W_0$.

**Corollary 5.8.** *Let $\pi = \pi^{(1)} \otimes \cdots \otimes \pi^{(r)} \in B(\Lambda_0)^{\otimes d} \otimes B(\Lambda_m)^{\otimes r-d}$, and write $w\Lambda_0 = f(\pi^{(d)})$ and $w'\Lambda_m = i(\pi^{(d+1)})$, for $w \in W/W_0$ and $w' \in W/W_m$ respectively. Then $\pi$ belongs to $B(d\Lambda_0 + (r-d)\Lambda_m)$ if and only if*

(a) *$f(\pi^{(k)}) \geq i(\pi^{(k+1)})$, for $1 \leq k < d$,*
(b) *$ww_0 \geq w'$,*
(c) *$f(\pi^{(k)}) \geq i(\pi^{(k+1)})$, for $d < k < r$.*

*Proof.* If $\pi$ belongs to $B(d\Lambda_0 + (r-d)\Lambda_m)$, then Theorem 5.7 gives a non-increasing sequence in $W$, which implies that conditions (a) to (c) hold.

Suppose that conditions (a) to (c) hold. Consider the elements $w \in W/W_0$ such that $w\Lambda_0$ appears as one of the direction vectors of $\pi^{(1)}, \dots, \pi^{(d)}$. Multiplying them with $w_0$ simultaneously, we can find the desired sequence in $W$. Thus, Theorem 5.7 implies that $\pi$ belongs to $B(d\Lambda_0 + (r-d)\Lambda_m)$. □

Our purpose is to interpret this result in terms of Young diagrams. To achieve this goal, we first have to find which partitions correspond to $f(\pi)$ and $i(\pi)$ when $\pi$ corresponds to a partition $\lambda$.

For this, we need to use the approach to path models in [K5] and [K1, chapter 8].

**Definition 5.9.** *Let $B$ and $B'$ be crystals. A map $\psi : B \to B'$ is called a* **crystal morphism of amplitude** *$h$ if*

(i) *$wt(\psi(b)) = hwt(b)$, $\epsilon_i(\psi(b)) = h\epsilon_i(b)$ and $\varphi_i(\psi(b)) = h\varphi_i(b)$,*
(ii) *$\psi(\tilde{e}_i b) = \tilde{e}_i^h \psi(b)$ and $\psi(\tilde{f}_i b) = \tilde{f}_i^h \psi(b)$, for all $b \in B$.*

**Definition 5.10.** (1) *$U_v^-(\mathfrak{g})$ is a module over the Kashiwara algebra, which defines a crystal. This is the crystal $B(\infty)$ and*

$$\epsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k b \neq 0\}, \;\; \varphi_i(b) = \epsilon_i(b) + wt(b)(h_i).$$

(2) *Define, for $a \in \mathbb{Z}$,*

$$wt(a) = a\alpha_i, \;\; \epsilon_j(a) = \begin{cases} -a & (j = i) \\ -\infty & (j \neq i) \end{cases}, \;\; \varphi_j(a) = \begin{cases} a & (j = i) \\ -\infty & (j \neq i) \end{cases}$$

*and*

$$\tilde{e}_j(a) = \begin{cases} a+1 & (j=i) \\ 0 & (j \neq i) \end{cases}, \quad \tilde{f}_j(a) = \begin{cases} a-1 & (j=i) \\ 0 & (j \neq i) \end{cases}.$$

*Then $\mathbb{Z}$ becomes a crystal. This is the crystal $B_i$.*

(3) *Let $\Lambda$ be a weight, and define*

$$wt(t_\Lambda) = \Lambda, \ \ \epsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty, \ \ \tilde{e}_i t_\Lambda = \tilde{f}_i t_\Lambda = 0.$$

*Then $\{t_\Lambda\}$ is the crystal $T_\Lambda$.*

(4) *Define another crystal structure on the underlying set of $B(\infty)$ by redefining $(wt, \epsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ by*

$$wt^{new} = -wt^{old}, \ \epsilon_i^{new} = \varphi_i^{old}, \ \varphi_i^{new} = \epsilon_i^{old}, \ \tilde{e}_i^{new} = \tilde{f}_i^{old}, \ \tilde{f}_i^{new} = \tilde{e}_i^{old}.$$

*This crystal is denoted by $B(-\infty)$. It may be considered as the crystal arising from the positive part $U_v^+(\mathfrak{g})$. We have*

$$\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k b \neq 0\}, \ \ \epsilon_i(b) = \varphi_i(b) - wt(b)(h_i).$$

We fix an infinite sequence $\underline{i} = (\cdots, i_k, \cdots, i_2, i_1)$ such that $i_k \neq i_{k+1}$, for all $k$, and that $i$ appears infinitely many times in the sequence, for all $i$. Then we can realize $B(\infty)$ as a subcrystal of $\mathbb{Z}^\infty_{\underline{i}} = \cdots \otimes B_{i_k} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1}$ [K3, Theorem 2.2.1]. This is the **Kashiwara embedding** and the **polyhedral realization** associated with $\underline{i}$.

**Proposition 5.11** ([K1, Proposition 8.1.3])**.** *For all $h \in \mathbb{N}$, there exists a unique crystal morphism $S_h : B(\infty) \to B(\infty)$ of amplitude $h$. $S_h$ is an injective map. In any polyhedral realization, we have*

$$S_h(\cdots, a_k, \cdots, a_2, a_1) = (\cdots, ha_k, \cdots, ha_2, ha_1).$$

In fact, this is proved by defining $S_h$ by the above formula in the polyhedral realization of $B(\infty)$ and showing that this is a crystal morphism of amplitude $h$. Define $S_h : B(\infty) \otimes T_\Lambda \to B(\infty) \otimes T_{h\Lambda}$ by $b \otimes t_\Lambda \mapsto S_h(b) \otimes t_{h\Lambda}$. This is again a crystal morphism of amplitude $h$.

**Proposition 5.12** ([K1, Corollary 8.1.5])**.** *Let $\Lambda$ be dominant integral. Then there exists a unique crystal morphism $S_h : B(\Lambda) \to B(h\Lambda)$ of amplitude $h$, for all $h \in \mathbb{N}$. Further, we have the following commutative diagram.*

$$\begin{array}{ccc} B(\Lambda) & \xrightarrow{S_h} & B(h\Lambda) \\ \cap & & \cap \\ B(\infty) \otimes T_\Lambda & \xrightarrow{S_h} & B(\infty) \otimes T_{h\Lambda} \end{array}$$

Let $\lambda \in B(\Lambda_m)$. Using the canonical embedding $B(h\Lambda_m) \subset B(\Lambda_m)^{\otimes h}$, we can write

$$S_h(\lambda) = \lambda^{(1)} \otimes \cdots \otimes \lambda^{(h)}.$$

We denote

$$S_h(\lambda)^{1/h} = {\lambda^{(1)}}^{\otimes 1/h} \otimes \cdots \otimes {\lambda^{(h)}}^{\otimes 1/h},$$

and replace $(\mu^{\otimes 1/h})^{\otimes k}$ with $\mu^{\otimes k/h}$, for any $\mu$ that appears in $\lambda^{(1)}, \ldots, \lambda^{(h)}$. In this way, we may write

$$S_h(\lambda)^{1/h} = {\nu_1}^{\otimes a_1} \otimes {\nu_2}^{\otimes (a_2 - a_1)} \otimes \cdots \otimes {\nu_s}^{\otimes (1-a_{s-1})},$$

where $a_0 = 0 < a_1 < \cdots < a_s = 1$ are rational integers and $\nu_1, \ldots, \nu_s$ are pairwise distinct $e$-restricted partitions. Then the following theorem holds.

**Theorem 5.13** ([K1, Proposition 8.3.2])**.** *If $h$ is sufficiently divisible then*
(1) $\nu_j = w_j\emptyset_m$, *for a unique* $w_j \in W/W_m$.
(2) $a_j$ *and* $\nu_j$ *all stabilize.*

**Theorem 5.14** ([K1, Proof of Theorem 8.2.3])**.** *Given sufficiently divisible $h$, we write*

$$S_h(\lambda)^{1/h} = {\nu_1}^{\otimes a_1} \otimes {\nu_2}^{\otimes(a_2-a_1)} \otimes \cdots \otimes {\nu_s}^{\otimes(1-a_{s-1})}$$

*as above, and define $\pi_\lambda$ to be the path given by $(wt(\nu_1), \ldots, wt(\nu_s); a_0, \ldots, a_s)$. Then $wt(\nu_j) = w_j\Lambda_m$, for $1 \le j \le s$, and the following hold.*
(1) $\pi_\lambda$ *is a LS-path.*
(2) *The map $B(\Lambda_m) \to \mathbb{B}(\Lambda_m)$ defined by $\lambda \mapsto \pi_\lambda$ is an isomorphism of crystals.*

The proof of [K1, Proposition 8.3.2] also gives a very explicit inductive algorithm to compute the $e$-cores $\nu_j$ as follows.

Recall that the tensor product rule for $B(\Lambda_m)^{\otimes r}$ is given by the following rule:

Let $\lambda^{(1)} \otimes \cdots \otimes \lambda^{(r)} \in B(\Lambda_m)^{\otimes r}$. Then, starting with $\lambda^{(r)}$, we read addable and removable $i$-nodes of each $\lambda^{(k)}$ from the first row to the last row, for $k = r, r-1, \ldots, 1$ succesively. We then apply the $RA$-deletion to the resulting sequence of dots, $A$'s and $R$'s.

**Lemma 5.15.** *Suppose that*

$$S_h(\lambda)^{1/h} = {\nu_1}^{\otimes a_1} \otimes {\nu_2}^{\otimes(a_2-a_1)} \otimes \cdots \otimes {\nu_s}^{\otimes(1-a_{s-1})}.$$

*and that $\tilde{f}_i\lambda \neq 0$. Then $(a_{k+1} - a_k)h$ are positive integers and we write*

$${\nu_1}^{\otimes a_1 h} \otimes {\nu_2}^{\otimes(a_2-a_1)h} \otimes \cdots \otimes {\nu_s}^{\otimes(1-a_{s-1})h} = \mu_1 \otimes \cdots \otimes \mu_h.$$

*Then we may write*

$$\tilde{f}_i^h(\mu_1 \otimes \cdots \otimes \mu_h) = \tilde{f}_i^{c_1}\mu_1 \otimes \tilde{f}_i^{c_2}\mu_2 \otimes \cdots \otimes \tilde{f}_i^{c_h}\mu_h,$$

*for some non-negative integers $c_j$ such that $\sum_{j=1}^h c_j = h$. Then, for some multiple $h'$ of $h$, we have*

$$S_{h'}(\tilde{f}_i\lambda)^{1/h'} = \big((s_i{\mu_1}^{\otimes(c_1/\varphi_i(\mu_1))h'/h} \otimes {\mu_1}^{\otimes(1-c_1/\varphi_i(\mu_1))h'/h}) \otimes \cdots$$
$$\cdots\cdots \otimes (s_i{\mu_h}^{\otimes(c_h/\varphi_i(\mu_h))h'/h} \otimes {\mu_h}^{\otimes(1-c_h/\varphi_i(\mu_h))h'/h})\big)^{1/h'}.$$

**Example 5.16.** *Let $m = 0$ and $e = 3$. Then $\lambda = (3, 1^2)$ is an $e$-core. Thus $S_h(\lambda)^{1/h} = (3, 1^2)$, for all $h$. Consider $\lambda' = (3, 1^3) = \tilde{f}_0\lambda$. Then, $\varphi_0(\lambda) = 3$ and we have, for $h$ which is divisible by* 3,

$$S_h(\lambda')^{1/h} = (4, 2, 1^2)^{\otimes 1/3} \otimes (3, 1^2)^{\otimes 2/3}.$$

**Definition 5.17.** *Suppose that*

$$S_h(\lambda)^{1/h} = {\nu_1}^{\otimes a_1} \otimes {\nu_2}^{\otimes(a_2-a_1)} \otimes \cdots \otimes {\nu_s}^{\otimes(1-a_{s-1})},$$

*for sufficiently divisible $h$. Then we call $\nu_1$ the* **ceiling** *of $\lambda$ and denote it by* $\mathrm{ceil}(\lambda)$. *Similarly, we call $\nu_s$ the* **floor** *of $\lambda$ and denote it by* $\mathrm{floor}(\lambda)$.

We have $wt(\mathrm{ceil}(\lambda)) = i(\lambda)$ and $wt(\mathrm{floor}(\lambda)) = f(\lambda)$ by the definitions and Theorem 5.14(2).

**Example 5.18.** *Let $m = 0$, $e = 3$, and $\lambda = (2^2, 1)$. Then, for $h$ which is divisible by 6,*

$$S_h(\lambda)^{1/h} = (5,3,1)^{\otimes 1/3} \otimes (4,2)^{\otimes 1/6} \otimes (2)^{\otimes 1/2}.$$

*Thus,* $\mathrm{ceil}(\lambda) = (5,3,1)$ *and* $\mathrm{floor}(\lambda) = (2)$.

Note that in this paper we define $\mathrm{ceil}(\lambda)$ in a different manner than [KLMW1], because we follow a slightly different line of proof. That the two definitions give the same $e$-core follows from Theorem 6.2 and Corollary 6.4 below, which prove Theorem 4.3, and Corollary 4.5.

Fayers pointed out that $\mathrm{ceil}(\lambda)$ and $\mathrm{floor}(\lambda)$ behave well under the Mullineux map. Let us review the Mullineux map quickly. Let $\mathcal{H}_n(q)$ be the Hecke algebra of type $A$. This is the $F$-algebra generated by $T_1, \dots, T_{n-1}$ subject to the quadratic relations $(T_i - q)(T_i + 1) = 0$ and the type $A$ braid relations. Let $\tau$ be the involution of $\mathcal{H}_n(q)$ defined by $T_i \mapsto qT_i^{-1}$. The simple $\mathcal{H}_n(q)$-modules are $\{D^\lambda \mid \lambda \text{ is } e\text{-restricted}\}$. Then the Mullineux map is defined by $D^{m(\lambda)} = (D^\lambda)^\tau$. In [LLT, Theorem 7.1], it is observed that the description of the Mullineux map obtained by Brundan and Kleshchev may be expressed in terms of the crystal $B(\Lambda_0)$. Shifting the residues, the Mullineux map may be described by $B(\Lambda_m)$ also.

**Proposition 5.19.** *Suppose that $\lambda \in B(\Lambda_m)$ is such that $\lambda = \tilde{f}_{m+i_1} \cdots \tilde{f}_{m+i_n} \emptyset$. Then we have $m(\lambda) = \tilde{f}_{m-i_1} \cdots \tilde{f}_{m-i_n} \emptyset$.*

**Corollary 5.20.** $\epsilon_{m+i}(\lambda) = \epsilon_{m-i}(m(\lambda))$ *and* $\varphi_{m+i}(\lambda) = \varphi_{m-i}(m(\lambda))$.

*Proof.* If $\tilde{e}^k_{m+i}\lambda \neq 0$ then $\tilde{e}^k_{m-i} m(\lambda) = m(\tilde{e}^k_{m+i}\lambda) \neq 0$. Thus we have $\epsilon_{m+i}(\lambda) \leq \epsilon_{m-i}(m(\lambda))$. Similarly, we have $\varphi_{m+i}(\lambda) \leq \varphi_{m-i}(m(\lambda))$. Then we also have $\epsilon_{m+i}(m(\lambda)) \leq \epsilon_{m-i}(\lambda)$ and $\varphi_{m+i}(m(\lambda)) \leq \varphi_{m-i}(\lambda)$. Hence the equalities hold. □

**Proposition 5.21.** *Let $\lambda \in B(\Lambda_m)$. Then* $\mathrm{ceil}(m(\lambda))$ *and* $\mathrm{floor}(m(\lambda))$ *are the conjugate partitions of* $\mathrm{ceil}(\lambda)$ *and* $\mathrm{floor}(\lambda)$ *respectively.*

*Proof.* We may assume that $m = 0$ without loss of generality. We prove by induction on $|\lambda|$ that if $S_h(\lambda) = \nu_1 \otimes \cdots \otimes \nu_h$ for sufficiently divisible $h$, then

$$S_h(m(\lambda)) = \nu'_1 \otimes \cdots \otimes \nu'_h,$$

where $\nu'_k$ is the conjugate partition of $\nu_k$, for all $k$.

If $|\lambda| = 0$ there is nothing to prove. Assume that the assertion holds for $\lambda$ and that $\tilde{f}_i\lambda \neq 0$. Note that $\nu_k$ are $e$-cores and thus $\nu_k$ has removable $i$-nodes only, or addable $i$-nodes only. If $\nu_k$ has $n_{i,k}$ removable $i$-nodes then $\nu'_k$ has $n_{i,k}$ removable $(-i)$-nodes, and similarly, if $\nu_k$ has $n_{i,k}$ addable $i$-nodes then $\nu'_k$ has $n_{i,k}$ addable $(-i)$-nodes. This implies that if

$$S_h(\tilde{f}_i\lambda) = \tilde{f}_i^{c_1}\nu_1 \otimes \cdots \otimes \tilde{f}_i^{c_h}\nu_h,$$

then

$$S_h(m(\tilde{f}_i\lambda)) = \tilde{f}_{-i}^{c_1}\nu'_1 \otimes \cdots \otimes \tilde{f}_{-i}^{c_h}\nu'_h.$$

Now, to obtain $S_{h'}(\tilde{f}_i\lambda)$, for sufficiently divisible $h'$, we replace $\tilde{f}_i^{c_k}\nu_k$ with

$$(s_i\nu_k)^{\otimes (c_k/\varphi_i(\nu_k))h'/h} \otimes \nu_k^{\otimes (1-c_k/\varphi_i(\nu_k))h'/h},$$

for all $k$. As $\varphi_i(\nu_k) = \varphi_{-i}(\nu'_k)$, the assertion holds for $\tilde{f}_i\lambda$. □

Recall that the Mullineax map is given by conjugation of a partition when $\mathcal{H}_n(q)$ is semisimple. The proof of Proposition 5.21 shows that the Mullineax map is always given by conjugation, if we work in the right model – the path model.

The descriptions of $\mathrm{ceil}(\lambda)$ and $\mathrm{floor}(\lambda)$ are a crucial part of our main results. In the case when $e = 2$, we have closed formulas for them.

**Proposition 5.22.** *Assume that $e = 2$ and that $\lambda \in B(\Lambda_m)$. Let $a(\lambda)$ be the length of the first row, and let $\ell(\lambda)$ be the length of the first column. Then*

$$\mathrm{ceil}(\lambda) = (\ell(\lambda), \ell(\lambda) - 1, \dots, 1), \quad \mathrm{floor}(\lambda) = (a(\lambda), a(\lambda) - 1, \dots, 1).$$

*Proof.* We prove both formulas by induction on the size of $\lambda$. As $\lambda$ is 2-restricted, the last node of the first column is removable. Let $i$ be its residue. Let $\mu = \tilde{e}_i^{max}\lambda = \tilde{e}_i^t\lambda$. Then by the induction hypothesis, we have

$$\mathrm{ceil}(\mu) = (\ell(\lambda) - 1, \ell(\lambda) - 2, \dots, 1).$$

Observe that there exists an addable normal $i$-node on the first column of $\mathrm{ceil}(\mu)$. Thus all normal $i$-nodes are addable and the addable $i$-node on the first column of $\mathrm{ceil}(\mu)$ is the first addable $i$-node to be changed into a removable $i$-node when $\tilde{f}_i^t$ is applied to $\mu$. Thus, Lemma 5.15 implies that

$$\mathrm{ceil}(\lambda) = s_i(\ell(\lambda) - 1, \ell(\lambda) - 2, \dots, 1) = (\ell(\lambda), \ell(\lambda) - 1, \dots, 1).$$

Hence, the formula for $\mathrm{ceil}(\lambda)$ is proved.

Next assume that the formula for $\mathrm{floor}(\lambda)$ is already proved. Consider the addable node on the first row. Let $i$ be its residue. Then, this addable node is a normal $i$-node. The induction hypothesis implies that $\mathrm{floor}(\lambda)$ has addable normal $i$-nodes. First suppose that $\varphi_i(\lambda) > 1$. Then $\tilde{f}_i\lambda$ differs from $\lambda$ at some node which lies in the second row or below. Thus $a(\tilde{f}_i\lambda) = a(\lambda)$. Let $h$ be sufficiently divisible. Then $\varphi_i(\lambda) > 1$ implies that we do not apply $\tilde{f}_i^{max} = \tilde{f}_i^{h\varphi_i(\lambda)}$ to $S_h(\lambda)$ when computing $S_h(\tilde{f}_i\lambda)$. Since the addable $i$-nodes of $\mathrm{floor}(\lambda)$ are the last addable normal $i$-nodes to be changed into removable $i$-nodes, that we do not apply $\tilde{f}_i^{max}$ to $S_h(\lambda)$ implies that $\mathrm{floor}(\tilde{f}_i\lambda) = \mathrm{floor}(\lambda)$ by Lemma 5.15. Hence we have proved the formula in this case. Second suppose that $\varphi_i(\lambda) = 1$. Then $\tilde{f}_i\lambda$ differs from $\lambda$ at the addable $i$-node on the first row. Thus $a(\tilde{f}_i\lambda) = a(\lambda)+1$. Let $h$ be sufficiently divisible. Then $\varphi_i(\lambda) = 1$ implies that we apply $\tilde{f}_i^{max}$ to $S_h(\lambda)$ when computing $S_h(\tilde{f}_i\lambda)$. As the addable $i$-nodes of $\mathrm{floor}(\lambda)$ are all normal, this implies that

$$\mathrm{floor}(\tilde{f}_i\lambda) = s_i\,\mathrm{floor}(\lambda) = s_i(a(\lambda), a(\lambda) - 1, \dots, 1) = (a(\lambda) + 1, a(\lambda), \dots, 1).$$

Hence, we have proved the formula in this case also. □

## 6. Description of Demazure crystals

**Lemma 6.1.**

(1) *Suppose that $i(\pi)(h_i) < 0$. Then*
  - (a) $\tilde{e}_i\pi \neq 0$.
  - (b) $i(\tilde{e}_i^{max}\pi) = s_i i(\pi) < i(\pi)$.

(2) *Suppose that $f(\pi)(h_i) > 0$. Then*
  - (a) $\tilde{f}_i\pi \neq 0$.
  - (b) $f(\tilde{f}_i^{max}\pi) = s_i f(\pi) > f(\pi)$.

*Proof.* (1) (a) $i(\pi)(h_i) < 0$ implies that $Q = \min\{\pi(t)(h_i) \mid 0 \leq t \leq 1\} < 0$. Thus, Lemma 5.6 implies that $Q \leq -1$ and $\tilde{e}_i\pi \neq 0$.

(b) By (a), $i(\tilde{e}_i^{max}\pi)(h_i) \geq 0$. Then

$$s_i i(\tilde{e}_i^{max}\pi) \geq i(\tilde{e}_i^{max}\pi).$$

On the other hand, we have either $i(\tilde{e}_i^{max}\pi) = i(\pi)$ or $s_i i(\pi)$. As $i(\pi)(h_i) < 0$ we have $s_i i(\pi) < i(\pi)$ and we have the result.

(2) (a) $f(\pi)(h_i) > 0$ implies that $Q \leq \pi(1)(h_i) - 1$ by the integrality condition again. Thus $\tilde{f}_i\pi \neq 0$.

(b) The proof is similar to that of (1). □

We thank Dr. Sagaki for showing us the proof of the following theorem. The proof for the first equality works for dominant integral weights in general.

**Theorem 6.2** ([L4, Theorem 2])**.** *Suppose $y \in W/W_m$. Then*

$$B_y(\Lambda_m) = \{\lambda \in B(\Lambda_m) \mid i(\lambda) \leq y\Lambda_m\} = \{\lambda \in B(\Lambda_m) \mid \mathrm{ceil}(\lambda) \subset y\emptyset_m\}.$$

*Proof.* We only have to prove the first equality. The second equality follows from the remark at the end of Definition 5.1 and Proposition 4.4. We prove

$$B_y(\Lambda_m) \supset \{\lambda \in B(\Lambda_m) \mid i(\lambda) \leq y\Lambda_m\}$$

by induction on $\ell(y)$. If $y = 1$ then $B_y(\Lambda_m) = \{\emptyset_m\}$ and $i(\lambda) \leq \Lambda_m$ implies that $\lambda = \emptyset_m$. Thus the statement is true.

Let $y = s_i s_{i_2} \cdots s_{i_\ell}$ be a reduced expression. First we remark that $s_i y\Lambda_m = y\Lambda_m$ is impossible: otherwise $s_i y = yu$, for some $u \in W_m$, which implies $\ell(s_i y) < \ell(y) \leq \ell(yu) = \ell(s_i y)$, a contradiction. Thus $y\Lambda_m(h_i) < 0$ and $s_i y \in W/W_m$.

Assume that $i(\lambda) \leq y\Lambda_m$. If $i(\lambda)(h_i) \geq 0$ then Lemma 5.2 (1) implies that $i(\lambda) \leq s_i y\Lambda_m$. Hence, by the induction hypothesis and the fact that $s_i y < y$, Proposition 4.2 (4) implies that $\lambda \in B_{s_i y}(\Lambda_m) \subset B_y(\Lambda_m)$. If $i(\lambda)(h_i) < 0$ then Lemma 6.1 (1) implies that $i(\tilde{e}_i^{max}\lambda) = s_i i(\lambda) < i(\lambda)$. Since $s_i y\Lambda_m < y\Lambda_m$ and $i(\lambda) \leq y\Lambda_m$, we have $s_i i(\lambda) \leq s_i y\Lambda_m$. The induction hypothesis then implies that $\tilde{e}_i^{max}\lambda \in B_{s_i y}(\Lambda_m)$. Now, $\lambda \in B_y(\Lambda_m)$ by Proposition 4.2 (2).

The opposite inclusion is easy to prove. In fact, if $\lambda \in B_y(\Lambda_m)$ then we may write $\lambda = \tilde{f}_i^{a_1} \cdots \tilde{f}_{i_\ell}^{a_\ell}\emptyset_m$. We apply $\tilde{f}_i^{a_1} \cdots \tilde{f}_{i_\ell}^{a_\ell}$ to the path associated with $\emptyset_m$ and we obtain $i(\lambda) = y'\Lambda_m$, for some $y' \leq y$. Hence $i(\lambda) \leq y\Lambda_m$. □

Theorem 4.3 is proved by the theorem below, which is called the "roof lemma" in [KLMW1].

**Theorem 6.3** ([KLMW1, Lemma 3.3])**.** *Let $\lambda \in B(\Lambda_m)$. Denote the residue of the removable node on the last row by $i$. Then*

$$\mathrm{roof}(\lambda) \supset \mathrm{roof}(\tilde{e}_i^{max}\lambda) = s_i\,\mathrm{roof}(\lambda).$$

**Corollary 6.4.** $\mathrm{roof}(\lambda) = \mathrm{ceil}(\lambda)$.

*Proof.* Note that Lemma 5.15 implies that

$$\mathrm{ceil}(\lambda) \supset \mathrm{ceil}(\tilde{e}_i^{max}\lambda) = s_i\,\mathrm{ceil}(\lambda).$$

Thus induction on the size of $\lambda$ proves the result. □

Hence, Theorem 4.3 follows from Theorem 6.2 and Corollary 6.4.

Our aim is to prove a similar result for $B^w(\Lambda_m)$. For this, we need a desciption of $B^w(\Lambda_m)$ which is similar to the description of $B_y(\Lambda_m)$ in Theorem 6.2. Fortunately, such a result exists. We thank Kashiwara and Sagaki, who kindly showed us the result. Here we follow Kashiwara's argument. As there exists no written proof, he permitted us to include his argument here.

Before explaining the result, which is Theorem 6.23 below, we recall more results from the crystal theory.

Denote the canonical basis of $U_v^-(\mathfrak{g})$ by $\{G_v(b) \mid b \in B(\infty)\}$. Let $\Lambda$ be a dominant integral weight. Then the irreducible highest weight module with highest weight $\Lambda$ has the basis $\{G_v(b)u_\Lambda \mid b \in B(\infty)\} \setminus \{0\}$, where $u_\Lambda$ is a highest weight vector. When $wt(G_v(b)u_\Lambda) = w\Lambda$, for $w \in W$, we denote $G_v(b)u_\Lambda$ by $u_{w\Lambda}$.

**Proposition 6.5** ([K3, Proposition 4.1])**.** (1) *Let $G_v(b)u_{w\Lambda} \neq 0$, for $b \in B(\infty)$. Then $G_v(b)u_{w\Lambda} = G_v(b')u_\Lambda$, for some $b' \in B(\infty)$.*
(2) *If $G_v(b)u_{w\Lambda} = G_v(b')u_{w\Lambda} \neq 0$, for $b, b' \in B(\infty)$, then $b = b'$.*

Let $(L_v(\Lambda), B(\Lambda))$ be the crystal basis of the integrable highest weight module $U_v(\mathfrak{g})u_\Lambda$. We have $\{G_v(b) \mid b \in B(\Lambda)\} = \{G_v(b)u_\Lambda \mid b \in B(\infty)\} \setminus \{0\}$. Then the following holds by [K3, (4.1)].

**Lemma 6.6.** *$\{G_v(b) \mid b \in B^w(\Lambda)\}$ is a basis of $U_v^-(\mathfrak{g})u_{w\Lambda}$.*

**Lemma 6.7.** *$\{G_v(b) \mid b \in B^w(\Lambda)\} = \{G_v(b)u_{w\Lambda} \mid b \in B(\infty)\} \setminus \{0\}$.*

*Proof.* Suppose that $b \in B^w(\Lambda)$. Then, Lemma 6.6 implies that we may write

$$G_v(b) = \sum_{b' \in B(\infty)} f_{b'} G_v(b')u_{w\Lambda},$$

for some $f_{b'} \in \mathbb{Q}(v)$. Then Proposition 6.5(1) asserts that each nonzero $G_v(b')u_{w\Lambda}$ is of the form $G_v(b")u_\Lambda$, for some $b" \in B(\infty)$. Therefore, $G_v(b) = G_v(b')u_{w\Lambda}$, for some $b' \in B(\infty)$.

Suppose that $G_v(b)u_{w\Lambda} \neq 0$, for some $b \in B(\infty)$. Then $G_v(b)u_{w\Lambda} = G_v(b')$, for some $b' \in B(\Lambda)$, by Proposition 6.5(1) again. Since

$$G_v(b') = G_v(b)u_{w\Lambda} \in U_v^-(\mathfrak{g})u_{w\Lambda},$$

Lemma 6.6 implies that $b' \in B^w(\Lambda)$. □

**Proposition 6.8.** *Assume that there exists a sequence*

$$w_1 \geq w_2 \geq \cdots \geq w_h = w.$$

*Then*

$$u_{w_1\Lambda} \otimes \cdots \otimes u_{w_h\Lambda} + vL_v(\Lambda)^{\otimes h} \in B^w(h\Lambda) \subset B(h\Lambda) \subset B(\Lambda)^{\otimes h}.$$

*Proof.* The proof is by induction on $h$. When $h = 1$, $u_{w\Lambda} + vL_v(\Lambda) \in B^w(\Lambda)$ by Lemma 6.7, so there is nothing to prove. Suppose that $h > 1$. By the induction hypothesis, we may assume that

$$u_{w_1\Lambda} \otimes \cdots \otimes u_{w_{h-1}\Lambda} + vL_v(\Lambda)^{\otimes(h-1)} \in B^{w_{h-1}}((h-1)\Lambda) \subset B^w((h-1)\Lambda).$$

This and Lemma 6.7 imply that there exists $b \in B(\infty)$ such that

$$u_{w_1\Lambda} \otimes \cdots \otimes u_{w_{h-1}\Lambda} + vL_v(\Lambda)^{\otimes(h-1)} = G_v(b)u_{(h-1)w\Lambda} + vL_v(\Lambda)^{\otimes(h-1)}.$$

Consider $G_v(b)(u_{hw\Lambda})$. As $G_v(b)(u_{hw\Lambda}) = 0$ or $G_v(b)(u_{hw\Lambda}) = G_v(b')u_{h\Lambda}$, for some $b' \in B(\infty)$, by Proposition 6.5(1), we have

$$G_v(b)(u_{(h-1)w\Lambda} \otimes u_{w\Lambda}) = G_v(b)(u_{hw\Lambda}) \in L_v(\Lambda)^{\otimes h}.$$

If we view $L_v(\Lambda)^{\otimes h}$ as a $\mathfrak{g}^{\otimes h}$-crystal lattice and consider its weight decomposition, $(G_v(b)u_{(h-1)w\Lambda}) \otimes u_{w\Lambda}$ is one of the weight components of $G_v(b)(u_{hw\Lambda})$. Thus

$$(G_v(b)u_{(h-1)w\Lambda}) \otimes u_{w\Lambda} \in L_v(\Lambda)^{\otimes h}.$$

As $(G_v(b)u_{(h-1)w\Lambda}) \otimes u_{w\Lambda} \notin vL_v(\Lambda)^{\otimes h}$ because

$$u_{w_1\Lambda} \otimes \cdots \otimes u_{w_{h-1}\Lambda} - G_v(b)u_{(h-1)w\Lambda} \in vL_v(\Lambda)^{\otimes(h-1)},$$

we may conclude that $G_v(b)(u_{hw\Lambda}) \neq 0$ and

$$u_{w_1\Lambda} \otimes \cdots \otimes u_{w_h\Lambda} + vL_v(\Lambda)^{\otimes h} = G_v(b)(u_{(h-1)w\Lambda} \otimes u_{w\Lambda}) + vL_v(\Lambda)^{\otimes h}.$$

On the other hand, Lemma 6.7 implies

$$G_v(b)(u_{(h-1)w\Lambda} \otimes u_{w\Lambda}) + vL_v(\Lambda)^{\otimes h} = G_v(b)u_{hw\Lambda} + vL_v(\Lambda)^{\otimes h} \in B^w(h\Lambda).$$

Thus we have proved

$$u_{w_1\Lambda} \otimes \cdots \otimes u_{w_h\Lambda} + vL_v(\Lambda)^{\otimes h} \in B^w(h\Lambda).$$

□

**Corollary 6.9.** *Let $w \in W$. If there exists a sequence $w_1 \geq w_2 \geq \cdots \geq w_h \geq w$ in $W$ such that $\nu_i = w_i\emptyset_m$, for $1 \leq i \leq h$, then*

$$\nu_1 \otimes \cdots \otimes \nu_h \in B^w(h\Lambda_m) \subset B(h\Lambda_m) \subset B(\Lambda_m)^{\otimes h}.$$

Define the $\mathbb{Q}(v)$-linear anti-involution $*$ on $U_v^-(\mathfrak{g})$ by $f_i^* = f_i$. It preserves the crystal lattice of $U_v^-(\mathfrak{g})$ [K2, Proposition 5.2.4]. Then, as in [K2, Corollary 6.1.2], $(G(b)^*, G(b)^*) \equiv (G(b), G(b)) \equiv 1$ modulo $v\mathbb{Z}[v]$ implies $G(b)^* = \pm G(b^*)$, for some $b^* \in B(\infty)$. Now, it is proved in [K3, Theorem 2.1.1] that the minus sign does not occur. To summarize, we have the following.

**Proposition 6.10.**

(1) *$B(\infty)^* = B(\infty)$.*
(2) *$G_v(b^*) = G_v(b)^*$, for $b \in B(\infty)$.*

Next let $\tilde{U}_v(\mathfrak{g})$ be the modified quantized enveloping algebra. Namely,

$$\tilde{U}_v(\mathfrak{g}) = \bigoplus_{\Lambda \in P} U_v(\mathfrak{g})a_\Lambda$$

such that $v^h a_\Lambda = a_\Lambda v^h = v^{\Lambda(h)}a_\Lambda$, $a_\Lambda e_i = e_i a_{\Lambda-\alpha_i}$, $a_\Lambda f_i = f_i a_{\Lambda+\alpha_i}$ and $a_\Lambda a_{\Lambda'} = \delta_{\Lambda\Lambda'}a_\Lambda$. Define the $\mathbb{Q}(v)$-linear anti-involution $*$ by

$$(v^h)^* = v^{-h}, \;\; e_i^* = e_i, \;\; f_i^* = f_i, \;\; a_\Lambda^* = a_{-\Lambda}.$$

Lusztig constructed global bases for tensor products of integrable highest weight and lowest weight $U_v(\mathfrak{g})$-modules [L, 24.3], and showed that their inverse limits exist in $\tilde{U}_v(\mathfrak{g})$. Thus we have the crystal basis of $\tilde{U}_v(\mathfrak{g})$ [L, 25.2]. We denote the crystal by

$$B(\tilde{U}_v(\mathfrak{g})) = \bigsqcup_{\Lambda \in P} B(U_v(\mathfrak{g})a_\Lambda).$$

The global basis of $\tilde{U}_v(\mathfrak{g})$ is also denoted by $\{G_v(b) \mid b \in B(\tilde{U}_v(\mathfrak{g}))\}$.

**Theorem 6.11** ([K4, Theorem 3.1.1])**.** *Let $\Lambda$ be an integral weight. We choose dominant integral weights $\Lambda^+$ and $\Lambda^-$ such that $\Lambda = \Lambda^+ - \Lambda^-$. Then combining two embeddings*

$$B(\Lambda^+) \subset B(\infty) \otimes T_{\Lambda^+}, \quad B(-\Lambda^-) \subset T_{-\Lambda^-} \otimes B(-\infty)$$

*and $T_{\Lambda^+} \otimes T_{-\Lambda^-} = T_\Lambda$, we have a strict embedding of crystals*

$$B(\Lambda^+) \otimes B(-\Lambda^-) \subset B(\infty) \otimes T_\Lambda \otimes B(-\infty).$$

*By taking the direct limit, we have*

$$B(U_v(\mathfrak{g})a_\Lambda) \simeq B(\infty) \otimes T_\Lambda \otimes B(-\infty).$$

In the remainder of this discussion, we identify $B(U_v(\mathfrak{g})a_\Lambda)$ with $B(\infty) \otimes T_\Lambda \otimes B(-\infty)$. The following theorem generalizes Proposition 6.10.

**Theorem 6.12** ([K4, Theorem 4.3.2, Corollary 4.3.3])**.**
(1) *$B(\tilde{U}_v(\mathfrak{g}))^* = B(\tilde{U}_v(\mathfrak{g}))$, and if $b = b_1 \otimes t_\Lambda \otimes b_2 \in B(\tilde{U}_v(\mathfrak{g}))$ then*

$$b^* = b_1^* \otimes t_{-\Lambda - wt(b_1) - wt(b_2)} \otimes b_2^*.$$

(2) *$G_v(b^*) = G_v(b)^*$, for $b \in B(\tilde{U}_v(\mathfrak{g}))$.*

Now we define, for $b \in B(\tilde{U}_v(\mathfrak{g}))$,

$$\epsilon_i^*(b) = \epsilon_i(b^*), \quad \varphi_i^*(b) = \varphi_i(b^*), \quad wt^*(b) = wt(b^*),$$
$$\tilde{e}_i^* b = (\tilde{e}_i b^*)^*, \quad \tilde{f}_i^* b = (\tilde{f}_i b^*)^*.$$

Then this defines another crystal structure on $B(\tilde{U}_v(\mathfrak{g}))$, which is called the star crystal structure. The star crystal structure is compatible with the original crystal structure on $B(\tilde{U}_v(\mathfrak{g}))$ in the following sense.

**Theorem 6.13** ([K4, Theorem 5.1.1])**.** *$\tilde{e}_i^*$ and $\tilde{f}_i^*$ are strict morphisms of crystals.*

Using the star crystal structure, we can define another Weyl group action on $B(\tilde{U}_v(\mathfrak{g}))$. We denote the action by $w^*b$, for $w \in W$ and $b \in B(\tilde{U}_v(\mathfrak{g}))$.

**Definition 6.14.** *Let $B$ be a normal crystal. An element $b \in B$ of weight $\Lambda$ is called* **extremal** *if there exists a subset $\{b_w\}_{w\in W}$ of $B$ such that*

(i) *$b_w = b$ if $w = 1$.*
(ii) *If $w\Lambda(h_i) \geq 0$ then $\tilde{e}_i b_w = 0$ and $\tilde{f}_i^{max} b_w = b_{s_i w}$.*
(iii) *If $w\Lambda(h_i) \leq 0$ then $\tilde{f}_i b_w = 0$ and $\tilde{e}_i^{max} b_w = b_{s_i w}$.*

When $B = B(\Lambda)$ for a dominant integral weight $\Lambda$, this is a natural crystal analogue of extremal weight vectors in the highest weight module $U_v(\mathfrak{g})u_\Lambda$.

**Lemma 6.15.** *Let $\Lambda$ be dominant integral. Define $b_w = u_{w\Lambda} + vL_v(\Lambda) \in B(\Lambda)$, for $w \in W$.*
(1) *The set of extremal elements of $B(\Lambda)$ coincides with $\{b_w\}_{w\in W}$.*
(2) *$\{u_{w\Lambda}\}_{w\in W}$ are extremal vectors. That is, we have the following.*
  (i) *If $w\Lambda(h_i) \geq 0$ then $e_i u_{w\Lambda} = 0$ and $f_i^{(w\Lambda(h_i))} u_{w\Lambda} = u_{s_i w\Lambda}$.*
  (ii) *If $w\Lambda(h_i) \leq 0$ then $f_i u_{w\Lambda} = 0$ and $e_i^{(-w\Lambda(h_i))} u_{w\Lambda} = u_{s_i w\Lambda}$.*
(3) *If $s_i w < w$ and $b \in B(\infty)$ satisfies $G_v(b)u_{w\Lambda} \neq 0$ then $\epsilon_i^*(b) = 0$.*
(4) *Suppose that $s_i w < w$ and $b \in B(\infty)$ satisfies $\epsilon_i^*(b) = 0$. Then*

$$G_v(b)u_{w\Lambda} + vL_v(\Lambda) = G_v(\tilde{f}_i^{*\,-w\Lambda(h_i)} b)u_{s_i w\Lambda} + vL_v(\Lambda).$$

*Proof.* (1) and (2) are well-known, and we only prove (3) and (4). Note that $s_iw < w$ implies $w\Lambda(h_i) \le 0$. Thus $f_iu_{w\Lambda} = 0$ by (2). If $\epsilon_i(b^*) > 0$ then $G_v(b^*) \in f_iU_v^-(\mathfrak{g})$. Thus $G_v(b) \in U_v^-(\mathfrak{g})f_i$ by Proposition 6.10. Then $G_v(b)u_{w\Lambda} = 0$, which contradicts our assumption. We have proved $\epsilon_i^*(b) = 0$.

To prove (4), note that $s_iw\Lambda(h_i) = -w\Lambda(h_i) \ge 0$ and $f_i^{(-w\Lambda(h_i))}u_{s_iw\Lambda} = u_{w\Lambda}$ by (2). Now, $\epsilon_i(b^*) = 0$ implies that

$$\tilde{f}_i^{-w\Lambda(h_i)}G_v(b^*) = f_i^{(-w\Lambda(h_i))}G_v(b^*).$$

Hence, we have

$$\big(\tilde{f}_i^{*\,-w\Lambda(h_i)}G_v(b)\big)u_{s_iw\Lambda} = G_v(b)f_i^{(-w\Lambda(h_i))}u_{s_iw\Lambda} = G_v(b)u_{w\Lambda}.$$

Thus $G_v(\tilde{f}_i^{*\,-w\Lambda(h_i)}b)u_{s_iw\Lambda} + vL_v(\Lambda) = G_v(b)u_{w\Lambda} + vL_v(\Lambda)$ follows. □

**Definition 6.16.** *Suppose that $\Lambda$ is dominant integral. For $w \in W$, we define*

$$B(w\Lambda) = \{b \in B(U_v(\mathfrak{g})a_{w\Lambda}) \mid b^* \text{ is extremal}\}.$$

We identify $B(w\Lambda)$ with a subcrystal of $B(\infty) \otimes T_{w\Lambda} \otimes B(-\infty)$ through the crystal isomorphism given in Theorem 6.11. As the property that $b^*$ is extremal is stable under $\tilde{e}_i$ and $\tilde{f}_i$, if we define $I_{w\Lambda}$ to be the subspace of $U_v(\mathfrak{g})a_{w\Lambda}$ spanned by $\{G_v(b) \mid b \notin B(w\Lambda)\}$ then it is a $U_v(\mathfrak{g})$-submodule of $U_v(\mathfrak{g})a_{w\Lambda}$. The $U_v(\mathfrak{g})$-module $V_v(w\Lambda) = U_v(\mathfrak{g})a_{w\Lambda}/I_{w\Lambda}$ is Kashiwara's extremal weight module.

**Theorem 6.17** ([K4, Proposition 8.2.2])**.** *Suppose that $\Lambda$ is dominant integral.*
(1) *$V_v(w\Lambda)$ is an integrable $U_v(\mathfrak{g})$-module.*
(2) *$B(w\Lambda)$ is the crystal graph of $V_v(w\Lambda)$.*
(3) *The map $b \mapsto w^*b$, for $b \in B(\Lambda)$, defines an isomorphism of crystals*

$$B(\Lambda) \simeq B(w\Lambda).$$

As $V_v(w\Lambda)$ is generated by the extremal vector of weight $w\Lambda$, and integrable, $V_v(w\Lambda)$ with $w = 1$ is the integrable highest weight module $U_v(\mathfrak{g})u_\Lambda$. Hence $B(w\Lambda)$ with $w = 1$ is nothing but $B(\Lambda)$, and there is no conflict in the notation.

Fix $\underline{i}$ and let $\mathbb{Z}_{\underline{i}}$ be the polyhedral realization of $B(\infty)$ as before. If $b \in B(\infty)$ corresponds to $(\cdots, 0, 0, a_r, \cdots, a_2, a_1) \in \mathbb{Z}_{\underline{i}}$, then the integers $a_k$ are determined by

$$b^* = \tilde{f}_{i_1}^{a_1}\tilde{f}_{i_2}^{a_2}\cdots\tilde{f}_{i_r}^{a_r}u_\infty$$

such that $\epsilon_{i_k}(\tilde{f}_{i_{k+1}}^{a_{k+1}}\tilde{f}_{i_{k+2}}^{a_{k+2}}\cdots u_\infty) = 0$, for all $k$. See [NZ, (2.35), (2.36)].

Define $S_h : B(\infty) \otimes T_\Lambda \otimes B(-\infty) \to B(\infty) \otimes T_{h\Lambda} \otimes B(-\infty)$ by

$$S_h(b_1 \otimes t_\Lambda \otimes b_2) = S_h(b_1) \otimes t_{h\Lambda} \otimes S_h(b_2).$$

This is also a crystal morphism of amplitude $h$.

The next results are proved in [NS1, Proposition 3.2, 3.5] in a slightly different manner.

**Lemma 6.18.**
(1) *Let $b \in B(\infty)$. Then $S_h(b)^* = S_h(b^*)$, for all $h$.*
(2) *Let $b \in B(\tilde{U}_v(\mathfrak{g}))$. Then $S_h(b)^* = S_h(b^*)$, for all $h$.*

*Proof.* (1) We fix a polyhedral realization $\mathbb{Z}_{\underline{i}}$ of $B(\infty)$ and denote by

$$(\ldots,0,0,a_r,\ldots,a_2,a_1)$$

the element which corresponds to $b$. Then $S_h(b)$ corresponds to

$$(\ldots,0,0,ha_r,\ldots,ha_2,ha_1)$$

by Proposition 5.11. Thus, we have

$$\begin{aligned}S_h(b)^* &= \tilde{f}_{i_1}^{ha_1}\tilde{f}_{i_2}^{ha_2}\cdots\tilde{f}_{i_r}^{ha_r}u_\infty = \tilde{f}_{i_1}^{ha_1}\tilde{f}_{i_2}^{ha_2}\cdots\tilde{f}_{i_r}^{ha_r}S_h(u_\infty)\\ &= \tilde{f}_{i_1}^{ha_1}\tilde{f}_{i_2}^{ha_2}\cdots\tilde{f}_{i_{r-1}}^{ha_{r-1}}S_h(\tilde{f}_{i_r}^{a_r}u_\infty) = \cdots\cdots = S_h(\tilde{f}_{i_1}^{a_1}\tilde{f}_{i_2}^{a_2}\cdots\tilde{f}_{i_r}^{a_r}u_\infty).\end{aligned}$$

Thus, $S_h(b)^* = S_h(b^*)$ as desired.

(2) Let $b = b'\otimes t_\Lambda\otimes b''$. Then $S_h(b)^*$ is equal to

$$\big(S_h(b')\otimes t_{h\Lambda}\otimes S_h(b'')\big)^* = S_h(b')^*\otimes t_{h(-\Lambda-wt(b')-wt(b''))}\otimes S_h(b'')^*.$$

Since $S_h(b^*) = S_h((b')^*)\otimes t_{h(-\Lambda-wt(b')-wt(b''))}\otimes S_h((b'')^*)$, $S_h(b)^* = S_h(b^*)$ follows by (1). □

**Lemma 6.19.** *Let $b\in B(\tilde{U}_v(\mathfrak{g}))$. If $b^*$ is extremal, so is $S_h(b)^*$.*

*Proof.* By the definition of tensor product, we have

$$\begin{aligned}\epsilon_i(b_1\otimes t_\Lambda\otimes b_2) &= \max(\epsilon_i(b_1), \epsilon_i(b_2)-(\Lambda+wt(b_1))(h_i)),\\ \varphi_i(b_1\otimes t_\Lambda\otimes b_2) &= \max(\varphi_i(b_1)+(\Lambda+wt(b_2))(h_i), \varphi_i(b_2)).\end{aligned}$$

Suppose that there exists $\{b_w = b'_w\otimes t_{-w\Lambda}\otimes b''_w\}_{w\in W}$ such that

(i) $b_w^* = (b'_w)^*\otimes t_{w\Lambda-wt(b'_w)-wt(b''_w)}\otimes(b''_w)^* = b^*$ if $w=1$.

(ii) If $w\Lambda(h_i)\geq 0$ then $\tilde{e}_i b_w^* = 0$ and $\tilde{f}_i^{max}b_w^* = b_{s_iw}^*$.

(iii) If $w\Lambda(h_i)\leq 0$ then $\tilde{f}_i b_w^* = 0$ and $\tilde{e}_i^{max}b_w^* = b_{s_iw}^*$.

We want to show that $\{S_h(b_w)^*\}_{w\in W}$ satisfies conditions (i) to (iii) above. As (i) is obvious, we prove (ii) and (iii). Suppose that $w\Lambda(h_i)\geq 0$. Then $\tilde{e}_i b_w^* = 0$ implies

$$\epsilon_i(b_w^*) = \max(\epsilon_i((b'_w)^*), \epsilon_i((b''_w)^*)+(-w\Lambda+wt(b''_w))(h_i)) = 0.$$

By Lemma 6.18, we have

$$\begin{aligned}\epsilon_i(S_h(b_w)^*) &= \epsilon_i(S_h((b'_w)^*)\otimes t_{h(w\Lambda-wt(b'_w)-wt(b''_w))}\otimes S_h((b''_w)^*))\\ &= \max(h\epsilon_i((b'_w)^*), h\epsilon_i((b''_w)^*)+h(-w\Lambda+wt(b''_w))(h_i)).\end{aligned}$$

Thus $\epsilon_i(S_h(b_w)^*) = h\epsilon_i(b_w^*) = 0$ and $\tilde{e}_iS_h(b_w)^* = 0$ follows. By a similar computation, we have $\varphi_i(S_h(b_w)^*) = h\varphi_i(b_w^*)$, which implies that

$$\begin{aligned}\tilde{f}_i^{max}S_h(b_w)^* &= \tilde{f}_i^{h\varphi_i(b_w^*)}S_h(b_w)^* = \tilde{f}_i^{h\varphi_i(b_w^*)}S_h(b_w^*)\\ &= S_h(\tilde{f}_i^{\varphi_i(b_w^*)}b_w^*) = S_h(b_{s_iw}^*) = S_h(b_{s_iw})^*.\end{aligned}$$

Suppose that $w\Lambda(h_i)\leq 0$. Then, by similar arguments, we have $\tilde{f}_iS_h(b_w)^* = 0$ and $\tilde{e}_i^{max}S_h(b_w)^* = S_h(b_{s_iw})^*$. □

Let $\Lambda$ be dominant integral, $w\in W$. Since $B(w\Lambda)\simeq B(\Lambda)$ by Theorem 6.17(3), we have a unique crystal morphism $B(w\Lambda)\to B(hw\Lambda)$ of amplitude $h$, which we also denote by $S_h$. The following corollary generalizes Proposition 5.12.

**Corollary 6.20.** *Let $\Lambda$ be dominant integral, $w \in W$. Then we have the following commutative diagram.*

$$\begin{array}{ccc} B(w\Lambda) & \xrightarrow{S_h} & B(hw\Lambda) \\ \cap & & \cap \\ B(\infty)\otimes T_{w\Lambda}\otimes B(-\infty) & \xrightarrow{S_h} & B(\infty)\otimes T_{hw\Lambda}\otimes B(-\infty) \end{array}$$

We need two formulas. In the lemma below, (1) is taken from [K4, (3.1.1)] and (2) is taken from [K6, Appendix].

**Lemma 6.21.**

(1) *Let $b = b_1 \otimes t_\Lambda \otimes b_2 \in B(\tilde{U}_v(\mathfrak{g}))$. Then $G_v(b) \in \tilde{U}_v(\mathfrak{g})$ equals $G_v(b_1)G_v(b_2)a_\Lambda$ plus the linear combination $\sum X_iY_ia_\Lambda$, where $X_i \in U_v^-(\mathfrak{g})_{-\alpha}$ and $Y_i \in U_v^+(\mathfrak{g})_\beta$ such that $ht(\alpha) < ht(wt(b_1))$ and $ht(\beta) < ht(wt(b_2))$ respectively. In particular,*

$$G_v(b_1 \otimes t_\Lambda \otimes u_{-\infty}) = G_v(b_1)a_\Lambda.$$

(2) *Let $b = b_1 \otimes t_\Lambda \otimes u_{-\infty}$ and suppose that $b^*$ is extremal. Then*

$$s_i^* b = \begin{cases} \tilde{f_i^*}^{-\Lambda(h_i)} b_1 \otimes t_{s_i\Lambda} \otimes u_{-\infty} & (\text{if } \epsilon_i^*(b) = 0.) \\ \tilde{e_i^*}^{max} b_1 \otimes t_{s_i\Lambda} \otimes \tilde{e_i^*}^{\Lambda(h_i)-\epsilon_i^*(b_1)} u_{-\infty} & (\text{if } \varphi_i^*(b) = 0.) \end{cases}$$

**Proposition 6.22.** *Suppose that $\Lambda$ is dominant integral.*

(1) *If $b \in B^w(\Lambda)$ then $w^*b \in B(\infty) \otimes t_{w\Lambda} \otimes u_{-\infty}$.*

(2) *Under the isomorphism $B(\Lambda) \simeq B(w\Lambda)$ given by $b \mapsto w^*b$, $B^w(\Lambda)$ may be identified with*

$$\{b \in B(\infty) \otimes t_{w\Lambda} \otimes u_{-\infty} \mid b^* \text{ is extremal}\}.$$

*Proof.* (1) We identify the extremal weight module $V_v(\Lambda)$ with the highest weight module $U_v(\mathfrak{g})u_\Lambda$ as before. Write $G_v(b) = G_v(b')u_\Lambda$ in $U_v(\mathfrak{g})u_\Lambda$. As

$$G_v(b' \otimes t_\Lambda \otimes u_{-\infty}) = G_v(b')a_\Lambda$$

by Lemma 6.21(1), we have $b = b' \otimes t_\Lambda \otimes u_{-\infty}$ under the identification of the crystal of the highest weight module $U_v(\mathfrak{g})u_\Lambda$ with $B(\Lambda)$ which is defined by the extremal weight module $V_v(\Lambda)$.

Suppose now that $b \in B^w(\Lambda)$. Then there exists $b_1 \in B(\infty)$ such that $G_v(b) = G_v(b_1)u_{w\Lambda}$ by Lemma 6.7. Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression. Then Lemma 6.15(3),(4) imply that

$$G_v(b) + vL_v(\Lambda) = G_v(\tilde{f_{i_\ell}^*}^{a_\ell} \cdots \tilde{f_{i_1}^*}^{a_1} b_1)u_\Lambda + vL_v(\Lambda),$$

where $a_k = -s_{i_k} \cdots s_{i_\ell}\Lambda(h_{i_k}) = s_{i_{k+1}} \cdots s_{i_\ell}\Lambda(h_{i_k})$, such that

$$\epsilon_{i_{k+1}}^*(\tilde{f_{i_k}^*}^{a_k} \cdots \tilde{f_{i_1}^*}^{a_1} b_1) = \epsilon_{i_{k+1}}(\tilde{f}_{i_k}^{a_k} \cdots \tilde{f}_{i_1}^{a_1} b_1^*) = 0,$$

for $0 \le k < \ell$. This implies $G_v(b) = G_v(\tilde{f_{i_\ell}^*}^{a_\ell} \cdots \tilde{f_{i_1}^*}^{a_1} b_1)u_\Lambda$. Thus, by the first paragraph, we have

$$b = \tilde{f_{i_\ell}^*}^{a_\ell} \cdots \tilde{f_{i_1}^*}^{a_1} b_1 \otimes t_\Lambda \otimes u_{-\infty}.$$

We show by downward induction on $k$ that

$$s_{i_{k+1}}^* \cdots s_{i_\ell}^* b = \tilde{f_{i_k}^*}^{a_k} \cdots \tilde{f_{i_1}^*}^{a_1} b_1 \otimes t_{s_{i_{k+1}} \cdots s_{i_\ell}\Lambda} \otimes u_{-\infty}.$$

If $k = \ell$ there is nothing to prove. Suppose that the equation holds for $k$. As $s_{i_{k+1}}^* \cdots s_{i_\ell}^* b \in B(s_{i_{k+1}} \cdots s_{i_\ell}\Lambda)$ by Theorem 6.17(3), $s_{i_{k+1}} \cdots s_{i_\ell} b^*$ is extremal. As

$$wt(s_{i_{k+1}} \cdots s_{i_\ell} b^*)(h_{i_k}) = -s_{i_{k+1}} \cdots s_{i_\ell}\Lambda(h_{i_k}) = -a_k \le 0,$$

we have $\varphi^*_{i_k}(s^*_{i_{k+1}}\cdots s^*_{i_\ell}b)=0$. Thus Lemma 6.21(2) implies

$$s^*_{i_k}\cdots s^*_{i_\ell}b=\tilde{e^*_{i_k}}^{max}\tilde{f^*_{i_k}}^{a_k}\cdots\tilde{f^*_{i_1}}^{a_1}b_1\otimes t_{s_{i_k}\cdots s_{i_\ell}\Lambda}\otimes \tilde{e^*_{i_k}}^{a_k-\epsilon_{i_k}(\tilde{f}^{a_k}_{i_k}\cdots\tilde{f}^{a_1}_{i_1}b^*_1)}u_{-\infty}.$$

Since the formula $\epsilon^*_{i_{k+1}}(\tilde{f^*_{i_k}}^{a_k}\cdots\tilde{f^*_{i_1}}^{a_1}b_1)=0$ implies $\epsilon_{i_k}(\tilde{f}^{a_k}_{i_k}\cdots\tilde{f}^{a_1}_{i_1}b^*_1)=a_k$ if we replace $k$ with $k-1$ in the formula, we have the equation for $k-1$. As a result, we have $w^*b=b_1\otimes t_{w\Lambda}\otimes u_{-\infty}\in B(\infty)\otimes t_{w\Lambda}\otimes u_{-\infty}$.

(2) We only have to show that if $b=b_1\otimes t_{w\Lambda}\otimes u_{-\infty}\in B(w\Lambda)$ then we have $(w^{-1})^*b\in B^w(\Lambda)$. Define $a_k=s_{i_{k+1}}\cdots s_{i_\ell}\Lambda(h_{i_k})$. We show by induction on $k$ that

$$s^*_{i_k}\cdots s^*_{i_1}b=\tilde{f^*_{i_k}}^{a_k}\cdots\tilde{f^*_{i_1}}^{a_1}b_1\otimes t_{s_{i_{k+1}}\cdots s_{i_\ell}\Lambda}\otimes u_{-\infty}.$$

If $k=0$ there is nothing to prove. Suppose that the equation holds for $k$. As $s_{i_k}\cdots s_{i_1}b^*$ is extremal and

$$wt(s_{i_k}\cdots s_{i_1}b^*)(h_{i_{k+1}})=-s_{i_{k+1}}\cdots s_{i_\ell}\Lambda(h_{i_{k+1}})=a_{k+1}\ge 0,$$

we have $\epsilon_{i_{k+1}}(s_{i_k}\cdots s_{i_1}b^*)=0$. Thus Lemma 6.21(2) implies the equation for $k+1$. As a result, we have

$$(w^{-1})^*b=\tilde{f^*_{i_\ell}}^{a_\ell}\cdots\tilde{f^*_{i_1}}^{a_1}b_1\otimes t_\Lambda\otimes u_{-\infty}.$$

Now, $\epsilon^*_{i_{k+1}}(\tilde{f^*_{i_k}}^{a_k}\cdots\tilde{f^*_{i_1}}^{a_1}b_1)=0$, for $0\le k<\ell$, because

$$\begin{aligned}0=\epsilon^*_{i_{k+1}}(s^*_{i_k}\cdots s^*_{i_1}b)&=\epsilon^*_{i_{k+1}}(\tilde{f^*_{i_k}}^{a_k}\cdots\tilde{f^*_{i_1}}^{a_1}b_1\otimes t_{s_{i_{k+1}}\cdots s_{i_\ell}\Lambda}\otimes u_{-\infty})\\&\ge\epsilon^*_{i_{k+1}}(\tilde{f^*_{i_k}}^{a_k}\cdots\tilde{f^*_{i_1}}^{a_1}b_1)\ge 0.\end{aligned}$$

Thus Lemma 6.15(4) shows that

$$G_v(b_1)u_{w\Lambda}=G_v(\tilde{f^*_{i_\ell}}^{a_\ell}\cdots\tilde{f^*_{i_1}}^{a_1}b_1)u_\Lambda=G_v((w^{-1})^*b).$$

Therefore, we have $(w^{-1})^*b\in B^w(\Lambda)$ by Lemma 6.7. □

The following is a theorem proved by Kashiwara and Sagaki independently. The proof for the first equality works for general dominant integral weights.

**Theorem 6.23.** *Suppose $w\in W/W_m$. Then*

$$B^w(\Lambda_m)=\{\lambda\in B(\Lambda_m)\mid f(\lambda)\ge w\Lambda_m\}=\{\lambda\in B(\Lambda_m)\mid \mathrm{floor}(\lambda)\supset w\emptyset_m\}.$$

*Proof.* If we write $\mathrm{floor}(\lambda)=u\emptyset_m$, for $u\in W/W_m$, then $f(\lambda)=wt(\mathrm{floor}(\lambda))=u\Lambda_m$, and $f(\lambda)\ge w\Lambda_m$ if and only if $u\ge w$. Thus, the second equality follows from Proposition 4.4. We prove the first equality.

Suppose that $h$ is sufficiently divisible and write $S_h(\lambda)=\nu_1\otimes\cdots\otimes\nu_h$, for $\lambda$ with $f(\lambda)\ge w\Lambda_m$. Then there exists a sequence $w_1\ge\cdots\ge w_h\ge w$ in $W$ such that $\nu_i=w_i\emptyset_m$, for $1\le i\le h$. By Corollay 6.9, we have $S_h(\lambda)\in B^w(h\Lambda_m)$. We want to show $\lambda\in B^w(\Lambda_m)$. Let us consider the crystal morphism of amplitude $h$:

$$B(\infty)\otimes T_{w\Lambda_m}\otimes B(-\infty)\longrightarrow B(\infty)\otimes T_{hw\Lambda_m}\otimes B(-\infty).$$

Then it induces $S_h:B(w\Lambda_m)\to B(hw\Lambda_m)$ by Corollary 6.20.

Write $w^*\lambda=b_1\otimes t_{w\Lambda_m}\otimes b_2\in B(w\Lambda_m)$. Note that we have $S_h(w^*\lambda)=w^*S_h(\lambda)$ by the uniqueness of the crystal morphism of amplitude $h$ given in Proposition 5.12. Since $S_h(\lambda)\in B^w(h\Lambda_m)$, we have

$$S_h(b_1)\otimes t_{hw\Lambda}\otimes S_h(b_2)=S_h(w^*\lambda)=w^*S_h(\lambda)\in B(\infty)\otimes t_{hw\Lambda_m}\otimes u_{-\infty}$$

by Proposition 6.22(2), which implies $S_h(b_2) = u_{-\infty}$. Since $S_h : B(\infty) \to B(\infty)$ is injective by Proposition 5.11, we have $w^*\lambda = b_1 \otimes t_{w\Lambda_m} \otimes u_{-\infty}$. Therefore, Proposition 6.22(2) implies that $\lambda \in B^w(\Lambda_m)$.

Next suppose that $\lambda \in B^w(\Lambda_m)$. Then, we have $S_h(\lambda) \in B^w(h\Lambda_m)$ by the similar argument. Take sufficiently divisible $h$ and write $S_h(\lambda) = \mu_1 \otimes \cdots \otimes \mu_h$, for $\mu_1 \geq \cdots \geq \mu_h$. Then $S_h(\lambda) \in B^w(h\Lambda_m)$ implies that

$$G_v(\mu_1 \otimes \cdots \otimes \mu_h) \in U_v^-(\mathfrak{g})(u_{w\Lambda_m} \otimes \cdots \otimes u_{w\Lambda_m}) \subset V_v(\Lambda_m) \otimes \cdots \otimes V_v(\Lambda_m).$$

Expand $G_v(\mu_1 \otimes \cdots \otimes \mu_h)$ in the basis $\{G_v(\nu_1) \otimes \cdots \otimes G_v(\nu_h) \mid \nu_1, \dots, \nu_h \in B(\Lambda_m)\}$. If $G_v(\nu_1) \otimes \cdots \otimes G_v(\nu_h)$ appears in the expansion then $\nu_1, \dots, \nu_h \in B^w(\Lambda_m)$, since

$$U_v^-(\mathfrak{g})(u_{w\Lambda_m} \otimes \cdots \otimes u_{w\Lambda_m}) \subset U_v^-(\mathfrak{g})u_{w\Lambda_m} \otimes \cdots \otimes U_v^-(\mathfrak{g})u_{w\Lambda_m}.$$

In particular, we have $\mu_1, \dots, \mu_h \in B^w(\Lambda_m)$. Write $\mu_h = y\emptyset_m$, for $y \in W/W_m$, and apply Proposition 4.2(4). Then $y \geq w$ and $f(\lambda) = wt(\mu_h) \geq w\Lambda_m$ follows. □

## 7. A property of Base

We write $\lambda \leq \mu$ for $\lambda \subset \mu$ in this and the next sections.

Let $\lambda \in B(\Lambda_m)$ be $\lambda = (\lambda_1, \lambda_2, \dots)$. We denote $\mu = (\lambda_2, \lambda_3, \dots)$ and write $\lambda = \{\lambda_1\} \cup \mu$. In this section we shall show $\mathrm{base}(\lambda) = \mathrm{base}(\{\lambda_1\} \cup \mathrm{base}(\mu))$.

**Definition 7.1.** *Let $J \subset \mathbb{Z}$ and $x \in \mathbb{Z}$. Then we denote $J \cap \mathbb{Z}_{\leq x}$ by $J_{\leq x}$.*

**Lemma 7.2.** *Let $\lambda \in B(\Lambda_m)$, $J$ the corresponding set of beta numbers of charge $m$, $j_0 = \max J$. Write $K = J_{\leq j_0 - 1}$. Define $t = \min\{i \geq 0 \mid \mathrm{down}^i(K) = \mathrm{base}(K)\}$.*

(1) *Suppose that $j_0 - e \notin J$. Then the partition associated with* $\mathrm{base}(K) \cup \{j_0\}$ *is e-restricted and* $\mathrm{base}(J) = \mathrm{base}(\mathrm{base}(K) \cup \{j_0\})$.

(2) *Suppose that $j_0 - e \in J$ and fix $0 \leq s \leq t$. If there exists no $0 \leq i < s$ such that*

$$j_0 - e = \min W(\mathrm{down}^i(K)) < \min U(\mathrm{down}^i(K)) \leq j_0 - 1,$$

*then* $\mathrm{down}^s(J) = \mathrm{down}^s(K) \cup \{j_0\}$. *Furthermore,*

  (i) *if $s < t$ then $U(\mathrm{down}^s(J)) \neq \emptyset$ and $\min U(\mathrm{down}^s(J)) = \min U(\mathrm{down}^s(K))$,*

  (ii) *if $s = t$ then the partition associated with* $\mathrm{base}(K) \cup \{j_0\}$ *is e-restricted and* $\mathrm{base}(J) = \mathrm{base}(\mathrm{base}(K) \cup \{j_0\})$.

*Proof.* Define $J_i = \mathrm{down}^i(K \cup \{j_0\})$ and $K_i = \mathrm{down}^i(K)$, for $0 \leq i \leq t$.

(1) We prove $J_i = K_i \cup \{j_0\}$, $j_0 - e \notin J_i$ and $\max K_i \leq j_0 - 1$ by induction on $i$. When $i = 0$, there is nothing to prove. Suppose that $0 < i < t$ and that the claim holds for $i$. We want to show that $J_{i+1} = K_{i+1} \cup \{j_0\}$, $j_0 - e \notin J_{i+1}$ and $\max K_{i+1} \leq j_0 - 1$. As $i < t$, we have $U(K_i) \neq \emptyset$ and

$$U(K_i) \subset U(J_i) = U(K_i \cup \{j_0\}) \subset U(K_i) \cup \{j_0\}.$$

As $\min U(K_i) \leq \max K_i \leq j_0 - 1$ we have $\min U(J_i) = \min U(K_i)$, which we denote by $p'$. Hence $p' \leq j_0 - 1$ and $p' - e \neq j_0 - e$, which implies $j_0 - e \notin J_{i+1}$. We show that $\min W(J_i) = \min W(K_i)$. Let $q' = \min W(J_i)$. As $q' \leq p' \leq j_0 - 1$ and $q' \in J_i = K_i \cup \{j_0\}$, we have $q' \in K_i$. If $q' = p'$ then $q' \in W(K_i)$. If $q' < p'$ then $q' + e \notin K_i$ because $q' + e \notin J_i$. Thus we also have $q' \in W(K_i)$. Suppose that there exists $p' - e < x < q'$ such that $x \in K_i$ and $x + e \notin K_i$. If $x + e \notin J_i$ then the minimality of $q'$ is contradicted. If $x + e \in J_i$ then $x \in J_i$ and $x + e = j_0$, which contradicts the induction hypothesis $j_0 - e \notin J_i$. We have proved $\min W(K_i) = \min W(J_i)$. Therefore, $\max K_{i+1} \leq \max K_i \leq j_0 - 1$ and

$$J_{i+1} = \mathrm{down}(J_i) = \mathrm{down}(K_i) \cup \{j_0\} = K_{i+1} \cup \{j_0\}.$$

Now, $J_t = \mathrm{base}(K) \cup \{j_0\}$ is associated with an $e$-restricted partition by Lemma 2.7(2), and $\mathrm{base}(J) = \mathrm{base}(\mathrm{base}(K) \cup \{j_0\})$ follows.

(2) We prove that $J_i = K_i \cup \{j_0\}$ and $\max K_i \le j_0 - 1$, for $0 \le i \le s$. Suppose that $0 < i < s$ and that the claim holds for $i$. As $U(K_i) \ne \emptyset$, we have $U(J_i) \ne \emptyset$ and

$$p' = \min U(J_i) = \min U(K_i) \le j_0 - 1$$

as before. Let $q' = \min W(J_i)$. By the same argument as in (1), we also have $q' \in W(K_i)$. Suppose that there is $p' - e < x < q'$ such that $x \in K_i$ and $x + e \notin K_i$. If $x + e \notin J_i$ then the minimality of $q'$ is contradicted. If $x + e \in J_i$ then $x + e = j_0$. Thus

$$j_0 - e = \min W(K_i) < q' \le p' = \min U(K_i) \le j_0 - 1,$$

which contradicts our assumption. Hence we have $\min W(K_i) = \min W(J_i)$ and $J_{i+1} = K_{i+1} \cup \{j_0\}$ follows. We also have $\max K_{i+1} \le \max K_i \le j_0 - 1$. By setting $i = s$, we obtain $\mathrm{down}^s(J) = \mathrm{down}^s(K) \cup \{j_0\}$.

If $s < t$ then $U(K_s) \ne \emptyset$ and we have $U(J_s) \ne \emptyset$ and $\min U(J_s) = \min U(K_s)$ by the same argument as above. If $s = t$ then $J_t = \mathrm{base}(K) \cup \{j_0\}$ is associated with an $e$-restricted partition and we have $\mathrm{base}(J) = \mathrm{base}(\mathrm{base}(K) \cup \{j_0\})$. □

Let $\lambda \in B(\Lambda_m)$, $J$, $j_0$, $K$ and $t$ as above.

In the rest of this section we assume that $j_0 - e \in J$ and that there exists $0 \le a < t$ such that $U(\mathrm{down}^a(J)) \ne \emptyset$ and

(i) $\mathrm{down}^i(J) = \mathrm{down}^i(K) \cup \{j_0\}$, for $0 \le i \le a$.
(ii) $p'' = \min U(\mathrm{down}^a(K))$ and $q'' = \min W(\mathrm{down}^a(K))$ satisfy

$$p'' = \min U(\mathrm{down}^a(J)), \quad q'' = j_0 - e < p'' \le j_0 - 1.$$

We also define $p' = \min U(\mathrm{down}^a(J))$ and $q' = \min W(\mathrm{down}^a(J))$. Note that $q'' = j_0 - e \notin W(\mathrm{down}^a(J))$ by $p' = p'' \ne q''$ and $\mathrm{down}^a(J) = \mathrm{down}^a(K) \cup \{j_0\}$. Hence, $q' \ne q''$ and $\mathrm{down}^{a+1}(J) \ne \mathrm{down}^{a+1}(K) \cup \{j_0\}$. More precisely, we have

$$\mathrm{down}^{a+1}(K) = (\mathrm{down}^{a+1}(J) \setminus \{j_0, j_0 - e\}) \cup \{q'\}.$$

Further, $q' > q''$ since $q' \le q''$ would imply $q' < p'$ and $q' \in W(\mathrm{down}^a(K))$, which contradicts $q'' = \min W(\mathrm{down}^a(K))$. Thus we must have

$$j_0 - e < q' \le p' \le j_0 - 1.$$

We also have $j_0 - e\mathbb{Z}_{\ge 0} \subset \mathrm{down}^a(J)$ and $U(\mathrm{down}^a(J)) = U(\mathrm{down}^a(K))$. In fact, by $j_0 - e \in \mathrm{down}^a(K) \subset \mathrm{down}^a(J)$ and $j_0 - e < p'$, $j_0 - ke \in \mathrm{down}^a(J)$, for $k \ge 1$. As $j_0 \in \mathrm{down}^a(J)$, we conclude that $j_0 - e\mathbb{Z}_{\ge 0} \subset \mathrm{down}^a(J)$. Then

$$U(\mathrm{down}^a(K)) \subset U(\mathrm{down}^a(J)) \subset U(\mathrm{down}^a(K)) \cup \{j_0\}$$

implies $U(\mathrm{down}^a(J)) = U(\mathrm{down}^a(K))$.

**Definition 7.3.** *Let $x \in J$.*

(1) *We define the* **runner index of** $x$, *which we denote by $r(x)$, by*

$$1 \le r(x) \le e \ \text{ and } \ x + e\mathbb{Z} = j_0 + r(x) + e\mathbb{Z}.$$

(2) *The* **layer level of** $x$, *which we denote by $\ell(x)$, is defined by*

$$\ell(x) = -\frac{\min\{z \in j_0 + e\mathbb{Z} | z \ge x\} - j_0}{e}.$$

The definitions are naturally understood on the abacus display which is adjusted by $j_0$. Namely, we display $J$ on the abacus in such a way that $j_0$ is on the rightmost runner. Then the runner index is 1 to $e$ from left to right, and $x$ is $\ell(x)$ rows higher than $j_0$ in this $j_0$-adjusted abacus display.

Define $b \geq 1$ by $b = \min\{i \geq 0 \mid \mathrm{base}(J) = \mathrm{base}(\mathrm{down}^a(J)) = \mathrm{down}^{a+i}(J)\}$, and, this time, we define

$$J_i = \mathrm{down}^i(\mathrm{down}^a(J)) \ \text{ and } \ K_i = \mathrm{down}^i(\mathrm{down}^a(K)),$$

for $0 \leq i \leq b$. We set $p'_i = \min U(J_i)$, $q'_i = \min W(J_i)$, for $0 \leq i < b$. Note that we have either $\ell(p'_i) = \ell(q'_i)$ or $\ell(p'_i) = \ell(q'_i) - 1$. We also define $p''_i = \min U(K_i)$ and $q''_i = \min W(K_i)$ if $U(K_i) \neq \emptyset$.

**Definition 7.4.** *We say that $0 \leq i < b$ is a* **reset point** *if $\ell(p'_i) = \ell(q'_i) = 0$.*

As $j_0 - e < q_0 \leq p_0 \leq j_0 - 1$, $i = 0$ is a reset point.

**Definition 7.5.** *$U$ is the set of indices $0 \leq i < b$ such that $\ell(q'_i) = \ell(p'_i)$.*

$U$ is also the set of indices $0 \leq i < b$ such that $r(q'_i) \leq r(p'_i)$. Now, we analyze the relationship between $J_i$ and $K_i$ in detail. We start with an example.

**Example 7.6.** *If $q' = p'$ and*

$$J_0 : \begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \times & & & \times & & \times \\ \times & & & & & \times \\ \times & & & & & \times \\ & & & & & \times \\ & \times & & & \times & \times \end{array} \qquad K_0 : \begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \times & & & \times & & \times \\ \times & & & & & \times \\ \times & & & & & \times \\ & & & & & \times \\ & \times & & & \times & \end{array}$$

*then $K_0 = J_0 \setminus \{j_0\}$ and $0 \in U$. We compute $J_i$ and $K_i$, for $i > 0$.*

$$J_1 : \begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \times & & & \times & & \times \\ \times & & & & & \times \\ \times & & & & & \times \\ & \times & & & & \times \\ & & & & \times & \times \end{array} \qquad K_1 : \begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \times & & & \times & & \times \\ \times & & & & & \times \\ \times & & & & & \times \\ & \times & & & & \\ & \times & & & \times & \end{array}$$

*Thus, $K_1 = (J_1 \setminus \{j_0, j_0 - e\}) \sqcup \{q'_0\}$ and $1 \in U$.*

$$J_2 : \begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \times & & & \times & & \times \\ \times & & & & & \times \\ \times & \times & & & & \times \\ & & & & & \times \\ & & & & \times & \times \end{array} \qquad K_2 : \begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \times & & & \times & & \times \\ \times & & & & & \times \\ \times & \times & & & & \\ & \times & & & & \\ & \times & & & \times & \end{array}$$

*Thus, $K_2 = (J_2 \setminus \{j_0, j_0 - e, j_0 - 2e\}) \sqcup \{q'_0, q'_1\}$ and $2 \in U$.*

$$
J_3 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times & \times & \times \\
\times\times & & \times \\
\times & & \times \\
 & & \times \\
 & & \times\times
\end{array}
\qquad\qquad
K_3 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times & \times & \times \\
\times\times & & \\
\times\times & & \\
\times & & \\
\times & & \times
\end{array}
$$

*Thus,* $K_3 = (J_3 \setminus \{j_0, j_0 - e, j_0 - 2e, j_0 - 3e\}) \sqcup \{q'_0, q'_1, q'_2\}$ *and* $3 \notin U$.

$$
J_4 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \times \\
\times & & \times \\
 & & \times \\
 & & \times\times
\end{array}
\qquad\qquad
K_4 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \\
\times\times & & \\
\times & & \\
\times & & \times
\end{array}
$$

*Thus,* $K_4 = (J_4 \setminus \{j_0, j_0 - e, j_0 - 2e, j_0 - 3e\}) \sqcup \{q'_0, q'_1, q'_2\}$. *Note that* $i = 4$ *is a reset point. We also have* $4 \in U$.

$$
J_5 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \times \\
\times & & \times \\
 & & \times\times \\
 & & \times
\end{array}
\qquad\qquad
K_5 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \\
\times\times & & \\
\times & & \times \\
 & & \times
\end{array}
$$

*Thus,* $K_5 = (J_5 \setminus \{j_0, j_0 - e, j_0 - 2e, j_0 - 3e\}) \sqcup \{q'_4, q'_1, q'_2\}$ *and* $5 \in U$.

$$
J_6 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \times \\
\times & & \times\times \\
 & & \times \\
 & & \times
\end{array}
\qquad\qquad
K_6 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \\
\times\times & & \times \\
 & & \times \\
 & & \times
\end{array}
$$

*Thus,* $K_6 = (J_6 \setminus \{j_0, j_0 - e, j_0 - 2e, j_0 - 3e\}) \sqcup \{q'_4, q'_5, q'_2\}$ *and* $6 \in U$.

$$
J_7 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \times\times \\
 & & \times\times \\
 & & \times \\
 & & \times
\end{array}
\qquad\qquad
K_7 : \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \times \\
\times & & \times \\
 & & \times \\
 & & \times
\end{array}
$$

*Thus,* $K_7 = (J_7 \setminus \{j_0, j_0 - e, j_0 - 2e, j_0 - 3e\}) \sqcup \{q'_4, q'_5, q'_6\}$ *and* $7 \in U$.

$$
J_8: \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times\times \\
\times & & \times\times \\
 & & \times\times \\
 & & \times \\
 & & \times
\end{array}
\qquad\qquad
K_8: \begin{array}{ccc}
\times\times & \times\times & \times\times \\
\times\times & & \times \\
\times\times & & \times \\
\times & & \times \\
 & & \times \\
 & & \times
\end{array}
$$

*We finish with* $K_8 = (J_8 \setminus \{j_0, j_0 - e, j_0 - 2e, j_0 - 3e, j_0 - 4e\}) \sqcup \{q'_4, q'_5, q'_6, q'_7\}$.

**Lemma 7.7.** *Define* $p'_b = p'_{b-1} - e$. *Then, for each* $0 \le i \le b$, *there exist* $m_i \ge 0$ *and* $x_0, \ldots, x_{m_i-1} \in K_i \setminus \{p'_i\}$ *such that* $U(J_i) = U(K_i)$ *and*

(a) $j_0 - e\mathbb{Z}_{\ge 0} \subset J_i$ *and* $\max J_i = j_0$.
(b) $K_i = (J_i \setminus \{j_0, j_0 - e, \ldots, j_0 - m_i e\}) \sqcup \{x_0, \ldots, x_{m_i-1}\}$.
(c) *If* $x \in J_i$ *is such that* $r(x) \le r(p'_i)$ *then* $x \notin U(J_i)$ *unless* $x = p'_i$.
(d) *If* $x \in K_i$ *is such that* $r(x) \le r(p'_i)$ *then* $x \notin U(K_i)$ *unless* $x = p'_i$.
(e) $\ell(x_k) = k$, *for* $0 \le k \le m_i - 1$.
(f) $r(p'_i) \ge r(x_0) \ge \cdots \ge r(x_{m_i-1})$.
(g) *If there exists* $x \in J_i$ *such that* $1 \le \ell(x) \le m_i$ *and* $r(p'_i) < r(x) < e$ *then* $(x + e\mathbb{Z}) \cap \mathbb{Z}_{\le j_0} \subset J_i$.
(h) *If* $j_0 - (k+1)e < x < x_k$, *for some* $0 \le k \le m_i - 1$, *then* $x \notin J_i$ *and* $x \notin K_i$.

*Further,* $1 \le m_1 \le \cdots \le m_b$.

*Proof.* $m_1 \le \cdots \le m_b$ follows from (a), (b) and (f) because $p''_i = p'_i \notin j_0 + e\mathbb{Z}$ implies that elements cannot be added to $K_i \cap (j_0 + e\mathbb{Z})$, only removed.

$i = 0$ is a reset point and we already know that the claims hold when $i = 0$; $m_0 = 0$ and (e), (f), (g) and (h) are vacant conditions. Let $i_1$ be a reset point and assume that the claims hold when $i \le i_1$. Let $i_2 \le b - 1$ be maximal such that $p'_i$ decreases in the interval $i_1 \le i \le i_2$. We showed in section 2 that $p'_{i+1} = p'_i - e$ for $i_1 \le i < i_2$ and that $p'_{i_2+1} > p'_{i_1}$ if $i_2 + 1 < b$. We show that the claims hold for $i_1 \le i \le i_2 + 1$ and $m_1 \le \cdots \le m_{i_2+1}$. If $i_2 + 1 < b$ then $i_2 + 1$ is a reset point because $\ell(p'_{i_2+1}) = 0$ by $p'_{i_2+1} > p'_{i_1}$ and $\ell(q'_{i_2+1}) = 0$ by (a) and (g) for $i = i_2 + 1$. The condition (g) for $i = i_2 + 1$ is not vacant since we already know $m_1 = 1$. We repeat this process until $b$ is reached.

As we will see in the proof below, three patterns appear in the interval $i_1 \le i \le i_2 + 1$. The first pattern occurs in the interval $i_1 \le i < i_1 + m_{i_1}$, thus it does not occur when $i_1 = 0$, and we reach $i = i_2 + 1$ when we are performing the second or the third pattern. We will show that $p'_{i_2+1} - ke \notin J_{i_2+1}$, for $1 \le k \le m_{i_2+1}$, when $i_2 + 1$ is a reset point. Hence, we may assume that $p'_{i_1} - ke \notin J_{i_1}$, for $1 \le k \le m_{i_1}$, when the first pattern occurs at $i = i_1$.

Let $i = i_1 + k$. When $k = 0$, $U(K_{i_1}) = U(J_{i_1}) \ne \emptyset$ and we have $x_0, \ldots, x_{m-1} \in K_{i_1} \setminus \{p'_{i_1}\}$ which satisfy (a) to (h), for $m = m_{i_1}$. We want to show that $i_1 + m \le b$ and that the claims hold for $i_1 \le i \le i_1 + m$. If $m = 0$ then there is nothing to prove. Suppose that $m > 0$ and $p'_{i_1} - je \notin J_{i_1}$, for $1 \le j \le m$. Then $x_0 \ne p'_{i_1}$ and (f) for $i = i_1$ imply that $r(x_k) < r(p'_{i_1})$, for $0 \le k \le m - 1$. We shall show the following $(\dot{a})$ to $(\dot{h})$, for $0 \le k \le m$, by induction on $k$.

$(\dot{a})$ $j_0 - e\mathbb{Z}_{\ge 0} \subset J_{i_1+k}$ and $\max J_{i_1+k} = j_0$.
$(\dot{b})$ $K_{i_1+k} = (J_{i_1+k} \setminus \{j_0, j_0 - e, \ldots, j_0 - me\}) \sqcup \{q'_{i_1}, \ldots, q'_{i_1+k-1}, x_k, \ldots, x_{m-1}\}$.
$(\dot{c})$ If $x \in J_{i_1+k}$ is such that $r(x) \le r(p'_{i_1})$ then $x \notin U(J_{i_1+k})$ unless $x = p'_{i_1+k}$.

($\dot{d}$) If $x \in K_{i_1+k}$ is such that $r(x) \le r(p'_{i_1})$ then $x \notin U(K_{i_1+k})$ unless $x = p'_{i_1+k}$.
($\dot{e}$) $\ell(q'_{i_1+j}) = j$, for $0 \le j \le k-1$.
($\dot{f}$) $r(p'_{i_1}) \ge r(q'_{i_1}) \ge \cdots \ge r(q'_{i_1+k-1}) \ge r(x_k) \ge \cdots \ge r(x_{m-1})$.
($\dot{g}$) If there exists $x \in J_{i_1+k}$ such that $1 \le \ell(x) \le m$ and $r(p'_{i_1}) < r(x) < e$ then $(x + e\mathbb{Z}) \cap J_{\le j_0} \subset J_{i_1+k}$.
($\dot{h}$) If $\ell(x) = j$ and $r(x) < r(q'_{i_1+j})$, for some $0 \le j \le k-1$, or $\ell(x) = j$ and $r(x) < r(x_j)$, for some $k \le j \le m-1$, then $x \notin J_{i_1+k}$ and $x \notin K_{i_1+k}$.

If $k \le m-1$ we also show $p'_{i_1} - je \notin J_{i_1+k}$, for $k+1 \le j \le m$, $i_1 + k < b$ and $p'_{i_1+k} = p'_{i_1} - ke$.

Before proving these claims, we explain that these imply the desired claims for $i_1 \le i \le i_1 + m$. First, $r(x_k) < r(p'_{i_1})$, for $0 \le k \le m-1$, implies $x_j \ne p'_{i_1+k}, p'_{i_1+k} - e$, for $k \le j \le m-1$. We also have $q'_{i_1+j} \ne p'_{i_1+k}, p'_{i_1+k} - e$, for $0 \le j \le k-1$. This follows from ($\dot{e}$) when $0 \le k \le m-1$ or $i_1 + m = b$, since $p'_{i_1+k} = p'_{i_1} - ke$ in these cases, and from $r(p'_{i_1+m}) > r(p'_{i_1}) \ge r(q'_{i_1+j})$ when $i_1 + m$ is a reset point. Second, if $i_1 + k < b$ then $U(J_{i_1+k}) = U(K_{i_1+k})$. In fact, if $p'_{i_1+k} = p'_{i_1+k-1} - e$ then $U(J_{i_1+k}) = U(K_{i_1+k}) = \{p'_{i_1+k}\}$ on runners $1, \dots, r(p'_{i_1})$ by ($\dot{b}$), ($\dot{c}$), ($\dot{d}$), $q'_{i_1+j} \ne p'_{i_1+k}, p'_{i_1+k} - e$, for $0 \le j \le k-1$, and $x_j \ne p'_{i_1+k}, p'_{i_1+k} - e$, for $k \le j \le m-1$. If $p'_{i_1+k} > p'_{i_1}$ then $U(J_{i_1+k}) = U(K_{i_1+k}) = \emptyset$ on runners $1, \dots, r(p'_{i_1})$ by ($\dot{c}$), ($\dot{d}$) and $r(p'_{i_1+k}) > r(p'_{i_1})$. $J_{i_1+k} = K_{i_1+k}$ on runners $r(p'_{i_1}) + 1, \dots, e-1$ by ($\dot{b}$) and ($\dot{f}$), and $U(J_{i_1+k}) = U(K_{i_1+k}) = \emptyset$ on runner $e$ by ($\dot{a}$), ($\dot{b}$) and ($\dot{f}$). Thus, $U(J_{i_1+k}) = U(K_{i_1+k})$ if $i_1 + k < b$. If $i_1 + m = b$ then the same proof shows that $U(K_b) = \emptyset$. (a) to (h) for $i_1 \le i \le i_1 + m - 1$ or $i = i_1 + m$ when $i_1 + m = b$ clearly follows from ($\dot{a}$) to ($\dot{h}$). When $i_1 + m$ is a reset point, $U(J_{i_1+m}) = U(K_{i_1+m})$ implies (c) and (d) for $i = i_1 + m$. The other parts of (a) to (h) are obvious.

Now we prove the claims. The claims hold when $k = 0$. Suppose that the claims hold for $k$ such that $0 \le k \le m-1$. Thus $p'_{i_1} - je \notin J_{i_1+k}$, for $k+1 \le j \le m$, $i_1 + k < b$ and $p'_{i_1+k} = p'_{i_1} - ke$. If $k+1 \le m-1$ then $p'_{i_1} - je \notin J_{i_1+k+1}$, for $k+2 \le j \le m$, and $p'_{i_1} - (k+1)e \in U(J_{i_1+k+1})$. Hence, $i_1 + k + 1 < b$ and $p'_{i_1+k+1} < p'_{i_1+k}$ implies $p'_{i_1+k+1} = p'_{i_1} - (k+1)e$.

As $p'_{i_1+k} - e = p'_{i_1} - (k+1)e < x < x_k$, for $x \in J_{i_1+k}$, implies $x \notin W(J_{i_1+k})$ by ($\dot{a}$), ($\dot{g}$) and ($\dot{h}$), we have $x_k \le q'_{i_1+k} \le p'_{i_1+k}$. As $\ell(p'_{i_1+k}) = k$ and $\ell(x_k) = k$, this implies

$$\ell(q'_{i_1+k}) = k \ \text{ and } \ r(q'_{i_1+k}) \le r(p'_{i_1}) < e.$$

Hence, ($\dot{a}$) and ($\dot{e}$) for $k+1$ follow.

$\ell(x_k) = \ell(q'_{i_1+k})$ and $x_k \le q'_{i_1+k}$ imply $r(x_k) \le r(q'_{i_1+k})$. As $r(x_{k+1}) \le r(x_k)$, we have $r(x_{k+1}) \le r(q'_{i_1+k})$. If $k = 0$ then we have proved ($\dot{f}$) for $k+1$. If $k \ge 1$ then we have to show $r(q'_{i_1+k}) \le r(q'_{i_1+k-1})$. Note that we have either $r(q'_{i_1+k-1}) = r(p'_{i_1+k-1})$ or $r(q'_{i_1+k-1}) < r(p'_{i_1+k-1})$ by ($\dot{f}$). If $r(q'_{i_1+k-1}) = r(p'_{i_1+k-1})$ then we have $r(q'_{i_1+j}) = r(p'_{i_1+j})$ and $\ell(q'_{i_1+j}) = \ell(p'_{i_1+j})$, for $0 \le j \le k-1$, by ($\dot{f}$). This implies that $q'_{i_1+j} = p'_{i_1+j}$, for $0 \le j \le k-1$. Thus, $J_{i_1+k-1}$ is obtained from $J_{i_1}$ by moving the bead $p'_{i_1}$ up to $p'_{i_1+k-1} = q'_{i_1+k-1}$. Hence,

$$q'_{i_1+k-1} - e \in J_{i_1+k} \ \text{ and } \ q'_{i_1+k-1} \notin J_{i_1+k}.$$

If $r(q'_{i_1+k-1}) < r(p'_{i_1+k-1})$ then $q'_{i_1+k-1} \in J_{i_1+k-1}$ implies $q'_{i_1+k-1} - e \in J_{i_1+k-1}$ by ($\dot{c}$) for $k-1$. Thus, $q'_{i_1+k-1} - e \in J_{i_1+k}$ and $q'_{i_1+k-1} \notin J_{i_1+k}$ follow again.

Therefore, $q'_{i_1+k-1} - e \in W(J_{i_1+k})$ and we conclude

$$q'_{i_1+k} \le q'_{i_1+k-1} - e.$$

Then, $\ell(q'_{i_1+k}) = \ell(q'_{i_1+k-1} - e)$ implies $r(q'_{i_1+k}) \le r(q'_{i_1+k-1} - e) = r(q'_{i_1+k-1})$. We have proved $(\dot{f})$ for $k+1$. As $r(q'_{i_1+k}) \le r(p'_{i_1})$, $(\dot{g})$ for $k+1$ also follow.

Now $U(J_{i_1+k}) = U(K_{i_1+k})$ implies $p'_{i_1+k} \in U(K_{i_1+k})$ and $p''_{i_1+k} = p'_{i_1+k}$. Hence it is clear that $(\dot{c})$ and $(\dot{d})$ for $k+1$ hold.

To show that $q''_{i_1+k} = x_k$, first suppose that

$$p'_{i_1+k} - e = p'_{i_1} - (k+1)e < x < j_0 - (k+1)e,$$

for $x \in K_{i_1+k}$. Then $x \in J_{i_1+k}$ by $(\dot{b})$ and $(\dot{f})$, and $x + e \in J_{i_1+k}$ by $(\dot{g})$. Using $(\dot{b})$ and $(\dot{f})$ again, we have $x + e \in K_{i_1+k}$ and $x \notin W(K_{i_1+k})$. If

$$j_0 - (k+1)e < x < x_k,$$

then $x \notin K_{i_1+k}$ by $(\dot{h})$, and $x \notin W(K_{i_1+k})$ again. We have proved that $q''_{i_1+k} \ge x_k$. To see that $q''_{i_1+k} = x_k$, it remains to show $x_k \in W(K_{i_1+k})$.

Note that $\ell(x_k) = \ell(p'_{i_1+k})$ and $r(x_k) \le r(p_{i_1}) = r(p'_{i_1+k})$ imply that

$$p'_{i_1+k} - e < x_k \le p'_{i_1+k}.$$

As $x_k \in K_{i_1+k}$, $x_k \in W(K_{i_1+k})$ follows when $x_k = p'_{i_1+k}$. If $x_k < p'_{i_1+k}$, we have to show $x_k + e \notin K_{i_1+k}$. It is clear when $k = 0$. Suppose $k \ge 1$ and $x_k + e \in K_{i_1+k}$. Thus $(\dot{h})$ implies $r(x_k + e) \ge r(q'_{i_1+k-1})$. On the other hand, $(\dot{f})$ implies $\ell(x_k + e) = \ell(q'_{i_1+k-1})$ and $r(x_k + e) \le r(q'_{i_1+k-1})$. Hence $x_k + e = q'_{i_1+k-1} \in J_{i_1+k-1}$ follows. As $r(x_k + e) \le r(p'_{i_1+k-1})$, we have either $x_k \in J_{i_1+k-1}$ or $x_k + e = p'_{i_1+k-1}$ by $(\dot{c})$ for $k-1$. By $(\dot{b})$ for $k-1$, $x_k \in J_{i_1+k-1}$ does not occur. $x_k + e = p'_{i_1+k-1}$ implies $x_k = p'_{i_1+k} \in J_{i_1+k}$, which contradicts $(\dot{b})$. Therefore, $x_k + e \notin K_{i_1+k}$. We have proved that $x_k \in W(K_{i_1+k})$, and $q''_{i_1+k} = x_k$ follows. In other words, we have proved

$$K_{i_1+k+1} = (K_{i_1+k} \setminus \{x_k\}) \sqcup \{p'_{i_1+k+1}\}.$$

By $J_{i_1+k+1} \sqcup \{q'_{i_1+k}\} = J_{i_1+k} \sqcup \{p'_{i_1+k+1}\}$ and (b), $K_{i_1+k+1}$ is equal to

$$(J_{i_1+k+1} \setminus \{j_0, \dots, j_0 - me\}) \sqcup \{q'_{i_1}, \dots, q'_{i_1+k-1}, q'_{i_1+k}, x_{k+1}, \dots, x_{m-1}\}.$$

We have proved $(\dot{b})$ for $k+1$.

Finally, to prove $(\dot{h})$ for $k+1$, we have to show that $x \notin J_{i_1+k+1}$ and $x \notin K_{i_1+k+1}$ when $\ell(x) = k$ and $r(x) < r(q'_{i_1+k})$. If $x \in J_{i_1+k+1}$ then $x \ne p'_{i_1+k} - e, q'_{i_1+k}$ implies $x \in J_{i_1+k}$ and $x + e \in J_{i_1+k}$ by $x \notin W(J_{i_1+k})$. However, $x + e \notin J_{i_1+k}$ if $k = 0$, and if $k \ge 1$ then $\ell(x + e) = k - 1$ and $r(x + e) < r(q'_{i_1+k}) \le r(q'_{i_1+k-1})$ imply $x + e \notin J_{i_1+k}$ by $(\dot{h})$. We have proved $x \notin J_{i_1+k+1}$. If $x \in K_{i_1+k+1}$ then $x \in J_{i_1+k+1}$ or $x = q'_{i_1+k}$ by $(\dot{b})$ for $k+1$. As both do not occur, $x \notin K_{i_1+k+1}$.

We have proved the desired claims for $i_1 \le i \le i_1 + m$. Note that we have also proved that $i_1, \dots, i_1 + m \in U$.

Define $m' \ge m$ by $m' = i_2 - i_1$ if $i_1 + m, \dots, i_2 \in U$, and by

$$i_1 + m, i_1 + m + 1, \dots, i_1 + m' \in U \ \text{ and } \ i_1 + m' + 1 \notin U,$$

otherwise. We want to show that the claims hold for $i_1 + m \le i \le i_1 + m' + 1$. To do this, we show, for $m \le k \le m' + 1$, that $p''_{i_1+j} = p'_{i_1+j}$, for $0 \le j \le k - 1$, and

($\ddot{a}$) $j_0 - e\mathbb{Z}_{\ge 0} \subset J_{i_1+k}$ and $\max J_{i_1+k} = j_0$.

($\ddot{b}$) $K_{i_1+k} = (J_{i_1+k} \setminus \{j_0, j_0 - e, \dots, j_0 - ke\}) \sqcup \{q'_{i_1}, \dots, q'_{i_1+k-1}\}$.

(c̈) If $x \in J_{i_1+k}$ is such that $r(x) \le r(p'_{i_1+k})$ then $x \notin U(J_{i_1+k})$ unless $x = p'_{i_1+k}$.

(d̈) If $x \in K_{i_1+k}$ is such that $r(x) \le r(p'_{i_1+k})$ then $x \notin U(K_{i_1+k})$ unless $x = p'_{i_1+k}$.

(ë) $\ell(q'_{i_1+j}) = j$, for $0 \le j \le k-1$.

(f̈) $r(p'_{i_1}) \ge r(q'_{i_1}) \ge \cdots \ge r(q'_{i_1+k-1})$.

(g̈) If there exists $x \in J_{i_1+k}$ such that $1 \le \ell(x) \le k$ and $r(p'_{i_1}) < r(x) < e$ then $(x + e\mathbb{Z}) \cap J_{\le j_0} \subset J_{i_1+k}$.

(ḧ) If $j_0 - (j+1)e < x < q'_{i_1+j}$, for some $0 \le j \le k-1$, then $x \notin J_{i_1+k}$ and $x \notin K_{i_1+k}$.

By the same argument as before, these claims imply the desired claims for $i_1 + m \le i \le i_1 + m' + 1$. Suppose that the claims hold for $k$ such that $m \le k \le m'$. Thus $p'_{i_1+k} = p_{i_1} - ke$ and, by definition, $i_1 + k < b$. $i_1 + k \in U$ implies $\ell(q'_{i_1+k}) = \ell(p'_{i_1+k}) = k$ and $r(q'_{i_1+k}) \le r(p'_{i_1}) < e$. Thus (ä) and (ë) for $k+1$ follow. If $m = 0$ and $k = m$ then (f̈) for $k+1$ is clear. Otherwise, $k \ge 1$ and we have either $r(q'_{i_1+k-1}) = r(p'_{i_1+k-1})$ or $r(q'_{i_1+k-1}) < r(p'_{i_1+k-1})$ by $i_1 + k - 1 \in U$. Now the rest of the proof is entirely similar to the previous one. The only difference is that we prove $q''_{i_1+k} = j_0 - (k+1)e$. To prove this, suppose that $p'_{i_1+k} - e < x \le p'_{i_1+k}$. As $J_{i_1+k} = K_{i_1+k}$ on runners $r(p'_{i_1}) + 1, \dots, e-1$, $x \in W(K_{i_1+k})$ implies $x \ge j_0 - (k+1)e$. As $j_0 - (k+1)e \in W(K_{i_1+k})$ by (b̈), we have $q''_{i_1+k} = j_0 - (k+1)e$.

Note that we have also proved $m_{i_1+k} = k$, for $m \le k \le m' + 1$.

If $i_1 + m' + 1 = b$ then we have finished the proof. Suppose $i_1 + m' + 1 < b$ and $i_1 + m' = i_2$. As $p'_{i_2+1} - e \notin J_{i_2+1}$ implies $p'_{i_2+1} - e \notin J_{i_1+m'}$, (g̈) for $k = m'$ implies $p'_{i_2+1} - je \notin J_{i_1+m'}$, and thus $p'_{i_2+1} - je \notin J_{i_2+1}$, for $1 \le j \le m'$. Let $x = p'_{i_2+1} - (m'+1)e$. Then $x + e \notin J_{i_1+m'}$ and $p'_{i_1+m'} - e < x < p'_{i_1+m'}$. Thus, if $x \in J_{i_1+m'}$ then $x \in W(J_{i_1+m'})$ and the minimality of $q'_{i_1+m'}$ is contradicted. Therefore, $p'_{i_2+1} - je \notin J_{i_2+1}$, for $1 \le j \le m' + 1 = m_{i_2+1}$.

To complete the proof of Lemma 7.7, we consider the case $i_1 + m' < i_2$. Write $x'_k = q'_{i_1+k}$, for $0 \le k \le m'$. We have

$$r(p'_{i_1}) \ge r(x'_0) \ge \cdots \ge r(x'_{m'})$$

by (f̈) for $k = m' + 1$. We show $i \notin U$ and the claims (A) to (C) below, for $i_1 + m' + 1 \le i \le i_2 + 1$. They hold when $i = i_1 + m' + 1$. Suppose that the claims hold for $i$ such that $i_1 + m' + 1 \le i \le i_2$. Thus $i \le i_2 \le b - 1$, $\ell(p'_i) \ge m' + 1$, $r(p'_i) = r(p'_{i_1})$, $i \notin U$ and

(A) $j_0 - e\mathbb{Z}_{\ge 0} \subset J_i$ and $\max J_i = j_0$.

(B) $K_i = (J_i \setminus \{j_0, j_0 - e, \dots, j_0 - (m'+1)e\}) \sqcup \{x'_0, \dots, x'_{m'}\}$.

(C) If $x \in \mathbb{Z}$ is such that $0 \le r(x) \le r(p'_{i_1})$ then $x \notin U(J_i)$ and $x \notin U(K_i)$ unless $x = p'_i$.

Note that (B) implies $p'_i \in U(K_i)$, and (A), (B), (C) imply $U(J_i) = U(K_i)$ and $p''_i = p'_i$.

As $i \notin U$, we have $p'_{i_1} - (i+1)e < q'_i < j_0 - (i+1)e$ and $q'_i < p'_i$ implies $q'_i - e \in J_i$ and $q'_i - e \in J_{i+1}$. Thus, if $i + 1 \le i_2$ then $q'_i - e \in W(J_{i+1})$ and it follows that

$$p'_{i+1} - e < q'_{i+1} \le j_0 - (i+2)e.$$

Hence, $i + 1 \notin U$.

(B) implies $q'_i \in W(K_i)$. Thus $q''_i \le q'_i$. Then,

$$p'_i - e = p''_i - e < q''_i < j_0 - (i+1)e$$

and (B) implies $q''_i \in W(J_i)$, which proves $q''_i = q'_i$. Therefore, (A), (B), (C) for $i+1$ follow.

We have proved $U(J_i) = U(K_i)$, $p'_i \ne x'_k$, for $0 \le k \le m'$, and (a), (b), (c), (d), (e), (f), for $i_1 + m' + 1 \le i \le i_2 + 1$. Now, $\ell(q'_i) = \ell(p'_i) + 1 \ge m' + 2$ implies that we do not touch the layer levels smaller than or equal to $m' + 1$ on runners $r(p'_{i_1}) + 1, \ldots, e-1$. Thus (g), for $i_1 + m' + 1 \le i \le i_2 + 1$, follow. Similarly, we do not touch the layer levels smaller than or equal to $m'$ on runners $1, \ldots, r(p'_{i_1})$. Thus (h), for $i_1 + m' + 1 \le i \le i_2 + 1$, follows.

If $i_2 + 1 = b$ then we have finished the proof. Suppose $i_2 + 1 < b$. Then $i_2 + 1$ is a reset point and $r(p'_{i_1}) < r(p'_{i_2+1}) < e$. Thus, (g) for $i = i_2$ implies $p'_{i_2+1} - je \notin J_{i_2+1}$, for $1 \le j \le m_{i_2+1}$, since $\ell(q'_{i_2}) \ge m' + 2$, $m_{i_2+1} = m_{i_2} = m' + 1$ and $p'_{i_2+1} - e \notin J_{i_2+1}$.

Now, the induction on $i$ works and we have proved the claims for $0 \le i \le b$. □

By Lemma 7.7, there exists $m \ge 1$ such that we may write

$$K_b = (J_b \setminus \{j_0 - ke \mid 0 \le k \le m\}) \cup \{x_k \mid 0 \le k \le m-1\},$$

where $r(x_k) \le r(p'_b)$, for $0 \le k \le m-1$. Consider $K_b \cup \{j_0\}$. As $j_1 \in K_b$ and $j_0 - j_1 \le e$, the partition associated with $K_b \cup \{j_0\}$ is $e$-restricted. Note that $U(J_b) = \emptyset$ and $U(K_b) = \emptyset$. Hence, $U(K_b \cup \{j_0\}) = \{j_0\}$ and explicit computation of $\mathrm{down}^k(K_b \cup \{j_0\})$, for $k \ge 0$, by using (a) to (h), shows that we obtain $\mathrm{down}^{k+1}(K_b \cup \{j_0\})$ from $\mathrm{down}^k(K_b \cup \{j_0\})$ by moving $x_k$ to $j_0 - (k+1)e$, for $0 \le k \le m-1$. Thus we end up with $\mathrm{base}(K_b \cup \{j_0\}) = J_b$. Therefore,

$$\mathrm{base}(\mathrm{base}(K) \cup \{j_0\}) = \mathrm{base}(K_b \cup \{j_0\}) = J_b = \mathrm{base}(J).$$

We have now proved the following proposition.

**Proposition 7.8.** *Let $\lambda \in B(\Lambda_m)$ and $J$ the corresponding set of beta numbers of charge $m$. Set $K = J_{\le j_0 - 1}$, where $j_0 = \max J$. Then, the partition associated with* $\mathrm{base}(K) \cup \{j_0\}$ *is $e$-restricted and we have*

$$\mathrm{base}(J) = \mathrm{base}(\mathrm{base}(K) \cup \{j_0\}).$$

Let $\lambda \in B(\Lambda_m)$ and $J$ the corresponding set of beta numbers of charge $m$. We delete the first row from $\lambda$ and we denote the resulting partition by $\mu$. Assume that $\mathrm{base}(\mu)$ is already computed. Then it is easy to compute $\mathrm{base}(\lambda)$ by using the above proposition. It gives us an efficient inductive definition of base and it is possible to generalize main results in section 8 to other types $A^{(2)}_{2n}$ and $D^{(2)}_{n+1}$.

**Corollary 7.9.** *Let $\lambda \in B(\Lambda_m)$ and $J$ the corresponding set of beta numbers of charge $m$. Let $j_0 > \cdots > j_r$ be the largest $r+1$ members of $J$. Define $J_{r+1} = J_{\le j_r - 1}$ and $J_k = \mathrm{base}(J_{k+1}) \cup \{j_k\}$, for $k = r, \ldots, 0$. Then the partition associated with $J_k$ is $e$-restricted and* $\mathrm{base}(J) = \mathrm{base}(J_0)$.

*Proof.* We show by downward induction on $k$ that $\max J_k = j_k$ and $\mathrm{base}(J_{\le j_k}) = \mathrm{base}(J_k)$. When $k = r+1$ there is nothing to prove. Suppose that the equations hold for $k+1$. Then, $j_{k+1} \in \mathrm{base}(J_{k+1})$ and $j_k - j_{k+1} \le e$ imply that the partition associated with $J_k$ is $e$-restricted. Now, by Proposition 7.8 and the induction hypothesis,

$$\begin{aligned}\mathrm{base}(J_{\le j_k}) &= \mathrm{base}(J_{\le j_{k+1}} \cup \{j_k\}) = \mathrm{base}(\mathrm{base}(J_{\le j_{k+1}}) \cup \{j_k\}) \\ &= \mathrm{base}(\mathrm{base}(J_{k+1}) \cup \{j_k\}) = \mathrm{base}(J_k).\end{aligned}$$

Thus, $\mathrm{base}(J) = \mathrm{base}(J_0)$ follows. □

## 8. Base Theorem

Let $\lambda \in B(\Lambda_m)$ and $J$ the set of beta numbers of charge $m$. Define

$$M_i(\lambda) = M_i(J) = \max\{x \in J \mid x + e\mathbb{Z} = i\}.$$

**Lemma 8.1.** *Let $\lambda \in B(\Lambda_m)$.*

(1) *If $M_i(\lambda) \le M_{i+1}(\lambda)$ then $M_i(\mathrm{down}(\lambda)) \le M_{i+1}(\mathrm{down}(\lambda))$. In particular, if $M_i(\lambda) \le M_{i+1}(\lambda)$ then $s_i \,\mathrm{base}(\lambda) \le \mathrm{base}(\lambda)$.*

(2) *If $\lambda$ is an $s_i$-core and $s_i\lambda \ge \lambda$ then*
   - (i) $\mathrm{down}(\lambda)$ *and* $\mathrm{down}(s_i\lambda)$ *are $s_i$-cores,*
   - (ii) $\mathrm{down}(s_i\lambda) = s_i \,\mathrm{down}(\lambda)$.

(3) *Suppose that $\lambda$ is an $s_i$-core and $s_i\lambda \ge \lambda$.*
   - (a) *If $s_i \,\mathrm{base}(\lambda) > \mathrm{base}(\lambda)$ then $\mathrm{base}(s_i\lambda) = s_i \,\mathrm{base}(\lambda) > \mathrm{base}(\lambda)$.*
   - (b) *If $s_i \,\mathrm{base}(\lambda) \le \mathrm{base}(\lambda)$ then $\mathrm{base}(s_i\lambda) = \mathrm{base}(\lambda)$.*

(4) *Suppose that $\lambda$ has an addable $i$-node on the first row, and that if we delete the first row then the resulting partition, which we denote by $\mu$, is an $e$-core.*
   - (a) *Suppose that $s_i\mu \ge \mu$. Then $\mathrm{base}(\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda) = \mathrm{base}(\lambda) < s_i \,\mathrm{base}(\lambda)$.*
   - (b) *Suppose that $s_i\mu \le \mu$. Then $\varphi_i(\lambda) = 1$ and*

$$\mathrm{base}(\tilde{f}_i^{\varphi_i(\lambda)}\lambda) = s_i \,\mathrm{base}(\lambda) > \mathrm{base}(\lambda).$$

*Proof.* (1) Let $J$ be the corresponding set of beta numbers of charge $m$, and define $p'$ and $q'$ as in the definition of $\mathrm{down}(J)$. Note that adding the bead $p' - e$ does not affect $M_i(\lambda)$ or $M_{i+1}(\lambda)$ because if $q' < p'$ then there exists a larger element $p'$ in $J$. Thus it suffices to study the effect of moving $q'$.

First suppose that $q' \ne M_{i+1}(\lambda)$. Then

$$M_{i+1}(\mathrm{down}(\lambda)) = M_{i+1}(\lambda) \ge M_i(\lambda) \ge M_i(\mathrm{down}(\lambda)).$$

The last inequality is an equality when $q' \ne M_i(\lambda)$. $M_i(\mathrm{down}(\lambda)) \le M_{i+1}(\mathrm{down}(\lambda))$ holds.

Second suppose that $q' = M_{i+1}(\lambda)$. In particular, $q'$ is on the $(i+1)^{th}$ runner. Note that $p'$ cannot be on the $i^{th}$ runner: if so then $p' \ge q'$ would imply $p' \ge q' + e - 1$ and

$$M_i(\lambda) \ge p' \ge q' + e - 1 > q' = M_{i+1}(\lambda),$$

which contradicts our assumption.

We shall show $q' - 1 \notin J$. Suppose on the contrary that $q' - 1 \in J$. If $q' = p'$ then $q' - 1 > p' - e$ and $q' - 1 + e \notin J$ by $M_i(\lambda) \le M_{i+1}(\lambda) = q'$. This implies that $q' - 1 \in W(J)$, which contradicts $q' = \min W(J)$. If $q' < p'$ then we also have $q' - 1 > p' - e$ and $q' - 1 + e \notin J$, since $p' - e = q' - 1$ would imply that $p'$ is on the $i^{th}$ runner. Hence we reach the contradiction $q' - 1 \in W(J)$ again. We have proved that $q' - 1 \notin J$.

Now we are ready to prove that $M_i(\mathrm{down}(\lambda)) \le M_{i+1}(\mathrm{down}(\lambda))$. Since $q' - 1 \notin J$ and $M_i(\lambda) \le q'$, we have $M_i(\mathrm{down}(\lambda)) \le q' - 1 - e$.

Suppose that $q' < p'$. Since $p' = \min U(J)$, we have $q' - e \in J$ and

$$M_{i+1}(\mathrm{down}(\lambda)) = q' - e > M_i(\mathrm{down}(\lambda))$$

follows. If $q' = p'$ then we have $M_{i+1}(\mathrm{down}(\lambda)) = q' - e$ by definition, and the result again follows. We have proved the first half of the claim.

Now, define a decreasing sequence of partitions

$$\lambda = \lambda^{(0)} > \cdots > \lambda^{(k)} > \cdots > \lambda^{(s)} = \mathrm{base}(\lambda)$$

by $\mathrm{down}(\lambda^{(k)}) = \lambda^{(k+1)}$, for $0 \le k < s$. Then, by repeated use of the first half of the claim, we have $M_i(\mathrm{base}(\lambda)) \le M_{i+1}(\mathrm{base}(\lambda))$. This implies that the $e$-core $\mathrm{base}(\lambda)$ does not have an addable $i$-node. Thus $s_i\,\mathrm{base}(\lambda) \le \mathrm{base}(\lambda)$.

(2) Note that $s_i\lambda$ is an $s_i$-core by Lemma 3.4(3). Let $J$ be the set of beta numbers of charge $m$ associated with $\lambda$. As $\lambda$ is an $s_i$-core, $p' = \min U(J)$ cannot be on the $i^{th}$ or the $(i+1)^{th}$ runners. Since $s_i\lambda$ is obtained from $\lambda$ by the rule given in Lemma 3.4, both contain $p'$, that is, $p' = \min U(J) = \min U(s_iJ)$. Let $q' = \min W(J)$. Then $q'$ for $s_i\lambda$ is given by

$$\min W(s_iJ) = \begin{cases} q' + 1 = M_i(\lambda) + 1 = M_{i+1}(s_i\lambda) & \text{if } q' + e\mathbb{Z} = i. \\ q' - 1 = M_{i+1}(\lambda) - 1 = M_i(s_i\lambda) & \text{if } q' + e\mathbb{Z} = i+1. \\ q' & \text{otherwise.} \end{cases}$$

To see this, note that if $x < q'$ is located on a runner different from the $i^{th}$ and the $(i+1)^{th}$ runners and if $x$ satisfies $x \in J$, $p' - e < x$ and $x + e \notin J$, then $x \notin W(J)$, which implies $x \notin W(s_iJ)$.

Suppose that $q' + e\mathbb{Z} = i$. Then $q' < p'$ and $q' = M_i(\lambda) \ge M_{i+1}(\lambda)$ implies that

$$M_{i+1}(\lambda) - 1 \le q' - e \le p' - e.$$

Thus $M_{i+1}(\lambda) - 1 \notin W(s_iJ)$ and there is no element of $W(s_iJ)$ on the $i^{th}$ runner. On the other hand, we have $q' + 1 \in W(s_iJ)$ and $\min W(s_iJ) = q' + 1$ follows.

If $q' + e\mathbb{Z} = i + 1$ then $q' < p'$, $q' = M_{i+1}(\lambda)$ and $\min W(s_iJ) = q' - 1$ is easy to see. Similarly, we have $\min W(s_iJ) = q'$ otherwise. Now it is clear that

$$(i)\ \ \mathrm{down}(\lambda) \text{ and } \mathrm{down}(s_i\lambda) \text{ are } s_i\text{-cores}, \quad (ii)\ \ \mathrm{down}(s_i\lambda) = s_i\,\mathrm{down}(\lambda).$$

(3) To prove (a) and (b), we consider two decreasing sequences

$$\lambda = \lambda^{(0)} > \cdots > \lambda^{(k)} > \cdots > \lambda^{(s)} = \mathrm{base}(\lambda)$$
$$s_i\lambda = \mu^{(0)} > \cdots > \mu^{(k)} > \cdots > \mu^{(t)} = \mathrm{base}(s_i\lambda)$$

where $\mathrm{down}(\lambda^{(k)}) = \lambda^{(k+1)}$, for $0 \le k < s$, and $\mathrm{down}(\mu^{(k)}) = \mu^{(k+1)}$, for $0 \le k < t$.

(a) We prove by induction on $k$ that

$$(i)\ \ \lambda^{(k)} \text{ and } \mu^{(k)} \text{ are } s_i\text{-cores}, \quad (ii)\ \ \mu^{(k)} = s_i\lambda^{(k)}, \quad (iii)\ \ s_i\lambda^{(k)} \ge \lambda^{(k)},$$

for $0 \le k \le \min(s, t)$. This implies the desired result. In fact, as $\lambda^{(k)}$ is an $e$-core if and only if $\mu^{(k)} = s_i\lambda^{(k)}$ is an $e$-core by Lemma 3.4(3), we must have $s = t$. Thus $\mathrm{base}(s_i\lambda) = s_i\,\mathrm{base}(\lambda)$ follows.

If $k = 0$ then the claim holds by the hypothesis. Suppose that the claim holds for $k$. Then (2) implies

(i) $\lambda^{(k+1)} = \mathrm{down}(\lambda^{(k)})$ and $\mu^{(k+1)} = \mathrm{down}(\mu^{(k)}) = \mathrm{down}(s_i\lambda^{(k)})$ are $s_i$-cores,
(ii) $\mu^{(k+1)} = \mathrm{down}(\mu^{(k)}) = \mathrm{down}(s_i\lambda^{(k)}) = s_i\,\mathrm{down}(\lambda^{(k)}) = s_i\lambda^{(k+1)}$.

If $M_i(\mathrm{down}(\lambda^{(k)})) < M_{i+1}(\mathrm{down}(\lambda^{(k)}))$ then (1) implies that $s_i\,\mathrm{base}(\lambda) \le \mathrm{base}(\lambda)$, contradicting the hypothesis. Thus, $M_i(\mathrm{down}(\lambda^{(k)})) \ge M_{i+1}(\mathrm{down}(\lambda^{(k)}))$ and this and (i) imply

(iii) $s_i\lambda^{(k+1)} \ge \lambda^{(k+1)}$.

(b) If $s_i\lambda^{(0)} = \lambda^{(0)}$ then the result is obvious. Suppose that $s_i\lambda^{(0)} > \lambda^{(0)}$. As $s_i\lambda^{(t)} \le \lambda^{(t)}$, the same induction argument as in (a) proves that there exists the maximal $1 \le k_0 \le t$ such that

$$(i)\ \ \lambda^{(k)} \text{ and } \mu^{(k)} \text{ are } s_i\text{-cores}, \quad (ii)\ \ \mu^{(k)} = s_i\lambda^{(k)}, \quad (iii)\ \ s_i\lambda^{(k)} > \lambda^{(k)},$$

for $0 \le k \le k_0 - 1$. Then (i) for $k = k_0 - 1$ and $s_i\lambda^{(k_0-1)} > \lambda^{(k_0-1)}$ imply

$$M_i(\lambda^{(k_0-1)}) > M_{i+1}(\lambda^{(k_0-1)}).$$

Applying (2) once more, we also have

$$(i)\ \ \lambda^{(k_0)} \text{ is an } s_i\text{-core}, \quad (ii)\ \ \mu^{(k_0)} = s_i\lambda^{(k_0)}.$$

Let $J$ be the set of beta numbers of charge $m$ associated with $\lambda^{(k_0-1)}$. Then, $\lambda^{(k_0-1)}$ and $\mu^{(k_0-1)}$ both have $p' = \min U(J) = \min U(s_iJ)$. Consider $q' = \min W(J)$. Assume that $q' + e\mathbb{Z} \ne i$. Then

$$M_{i+1}(\lambda^{(k_0-1)}) < M_i(\lambda^{(k_0-1)}) = M_i(\lambda^{(k_0)}) \le M_{i+1}(\lambda^{(k_0)})$$

and $M_{i+1}(\lambda^{(k_0)})$ is either $M_{i+1}(\lambda^{(k_0-1)}) - e$ or $M_{i+1}(\lambda^{(k_0-1)})$. In either case, we have a contradiction, and we conclude that $q' + e\mathbb{Z} = i$. Then

$$M_{i+1}(\lambda^{(k_0-1)}) < M_i(\lambda^{(k_0-1)}) = M_i(\lambda^{(k_0)}) + e \le M_{i+1}(\lambda^{(k_0)}) + e$$

and $M_{i+1}(\lambda^{(k_0)}) + e = M_{i+1}(\lambda^{(k_0-1)}) + e$.

As $M_{i+1}(\lambda^{(k_0-1)}) < M_i(\lambda^{(k_0-1)})$ implies $M_{i+1}(\lambda^{(k_0-1)}) + e - 1 \le M_i(\lambda^{(k_0-1)})$ and $M_i(\lambda^{(k_0)}) \ne M_{i+1}(\lambda^{(k_0)})$, we have $M_i(\lambda^{(k_0)}) + 1 = M_{i+1}(\lambda^{(k_0)})$. Since $\lambda^{(k_0)}$ is also an $s_i$-core, this implies $s_i\lambda^{(k_0)} = \lambda^{(k_0)}$. Hence, we have $\mu^{(k_0)} = \lambda^{(k_0)}$, which implies $\mathrm{base}(s_i\lambda) = \mathrm{base}(\lambda)$.

(4) (a) Since $s_i\mu \ge \mu$, $\lambda$ does not have a removable $i$-node. Let $J$ be the set of beta numbers associated with $\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda$, and let $K$ be the set of beta numbers associated with $\lambda$. We have $\max J = j_0 = \max K$. Then

(i) By deleting the first row from $\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda$, we obtain $\tilde{f}_i^{max}\mu = s_i\mu$.
(ii) The set of beta numbers associated with $\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda$ is $s_i(J \setminus \{j_0\}) \cup \{j_0\}$.

If $\varphi_i(\lambda) = 1$ then the claim $\mathrm{base}(\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda) = \mathrm{base}(\lambda)$ is obvious. Assume that $\varphi_i(\lambda) > 1$. Then $\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda$ has both an addable $i$-node and a removable $i$-node, thus it cannot be an $e$-core. This implies $U(J) \ne \emptyset$ and we have $U(J) = \{j_0\}$, $p' = \min U(J) = j_0$.

Note that the abacus displays of $J$ and $K$ have the following form by (i) and (ii) above.

$$
J:\ \begin{array}{cccc}
\cdots & \times & \times & \cdots \\
\cdots & \times & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & & \cdots \\
\cdots & j_0 & & \cdots
\end{array}
\qquad
K:\ \begin{array}{cccc}
\cdots & \times & \times & \cdots \\
\cdots & \times & \times & \cdots \\
\cdots & \times & & \cdots \\
\cdots & \times & & \cdots \\
\cdots & \times & & \cdots \\
\cdots & & & \cdots \\
\cdots & j_0 & & \cdots
\end{array}
$$

Thus, there exists $k_0$ such that, for $0 \leq k \leq k_0$, $\mathrm{down}^{k+1}(\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda)$ and $\mathrm{down}^{k+1}(\lambda)$ are obtained from $\mathrm{down}^{k}(\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda)$ and $\mathrm{down}^{k}(\lambda)$ by moving the maximal element of the $j^{th}$ runner, for some $j \neq i, i+1$, to the $i^{th}$ runner, respectively. Note that $j$ is the same for $\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda$ and $\lambda$ in each step $k$. At $k = k_0$, we reach the following form.

$$
\mathrm{down}^{k_0}(J):\ \begin{array}{cccc}
\cdots & \times & \times & \cdots \\
\cdots & \times & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & \times & & \cdots \\
\cdots & j_0 & & \cdots
\end{array}
\qquad
\mathrm{down}^{k_0}(K):\ \begin{array}{cccc}
\cdots & \times & \times & \cdots \\
\cdots & \times & \times & \cdots \\
\cdots & \times & & \cdots \\
\cdots & \times & & \cdots \\
\cdots & \times & & \cdots \\
\cdots & \times & & \cdots \\
\cdots & j_0 & & \cdots
\end{array}
$$

Note that $\mathrm{down}^{k_0}(K) = \mathrm{base}(\lambda)$. In particular, we have $s_i\,\mathrm{base}(\lambda) > \mathrm{base}(\lambda)$. By computing $\mathrm{down}^{k}(J)$, for $k > k_0$, we conclude that $\mathrm{base}(\tilde{f}_i^{\varphi_i(\lambda)-1}\lambda) = \mathrm{base}(\lambda)$.

(b) Since $s_i\mu \leq \mu$, $\lambda$ has the unique addable $i$-node, which is the addable $i$-node on the first row. Thus, $\varphi_i(\lambda) = 1$ and we compare $\mathrm{base}(\tilde{f}_i\lambda)$ and $\mathrm{base}(\lambda)$. Let $J$ and $K$ be the corresponding sets of beta numbers, respectively. Then the abacus displays of $J$ and $K$ have the following form, where $j_0' = j_0 - 1$.

$$
J:\ \begin{array}{cccc}
\cdots & \times & \times & \cdots \\
\cdots & \times & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & & \cdots \\
\cdots & & j_0 & \cdots
\end{array}
\qquad
K:\ \begin{array}{cccc}
\cdots & \times & \times & \cdots \\
\cdots & \times & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & & \cdots \\
\cdots & j_0' & & \cdots
\end{array}
$$

By a similar argument as above, there exists $k_0$ such that $\mathrm{down}^{k_0}(J)$ and $\mathrm{down}^{k_0}(K)$ have the following form.

$$
\mathrm{down}^{k_0}(J):\ \begin{array}{cccc}
\cdots & \times & \times & \cdots \\
\cdots & \times & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & j_0 & \cdots
\end{array}
\qquad
\mathrm{down}^{k_0}(K):\ \begin{array}{cccc}
\cdots & \times & \times & \cdots \\
\cdots & \times & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & & \times & \cdots \\
\cdots & \times & & \cdots \\
\cdots & j_0' & & \cdots
\end{array}
$$

Thus, $\mathrm{base}(J) = \mathrm{down}^{k_0}(J)$, and by computing $\mathrm{down}^k(K)$, for $k > k_0$, we have $\mathrm{base}(J) = s_i \,\mathrm{base}(K) > \mathrm{base}(K)$. □

**Lemma 8.2.** *Let $\lambda \in B(\Lambda_m)$ and $J$ the corresponding set of beta numbers of charge $m$. Suppose that $\tilde{f}_i\lambda \neq 0$ and that $\tilde{f}_i J$ is obtained from $J$ by moving $x$ to $x+1$.*

(1) $s_i \,\mathrm{base}(J_{\le x-1}) \le \mathrm{base}(J_{\le x-1})$.
(2) $\mathrm{base}((\tilde{f}_i J)_{\le x+1}) = s_i \,\mathrm{base}(J_{\le x+1}) > \mathrm{base}(J_{\le x+1})$.
(3) *Suppose that $\{z \in J_{\ge x+1} \mid z + e\mathbb{Z} = i\} \neq \emptyset$. We denote*

$$y = \min\{z \in J_{\ge x+1} \mid z + e\mathbb{Z} = i\}.$$

*Then we have either*
 (i) $\mathrm{base}((\tilde{f}_i J)_{\le y-1}) = s_i \,\mathrm{base}(J_{\le y-1}) > \mathrm{base}(J_{\le y-1})$, *or*
 (ii) $\mathrm{base}((\tilde{f}_i J)_{\le y-1}) = \mathrm{base}(J_{\le y-1})$.

*Proof.* (1) Since $\tilde{f}_i J$ is obtained from $J$ by moving $x$ to $x+1$, $x$ is the smallest addable $i$-integer which corresponds to a normal $i$-node. Note that all the elements in

$$\{x - ke \in J \mid k \in \mathbb{Z}_{\ge 1},\ x - ke + 1 \notin J,\ x - ke > M_{i+1}(J_{\le x-1})\}$$

correspond to addable normal $i$-nodes. Thus, it must be empty and we have

$$M_i(J_{\le x-1}) \le M_{i+1}(J_{\le x-1}).$$

Now Lemma 8.1(1) implies the result.

(2) Note that $J_{\le x-1} = (\tilde{f}_i J)_{\le x-1}$ and

$$J_{\le x+1} = J_{\le x-1} \cup \{x\},\ \ (\tilde{f}_i J)_{\le x+1} = (\tilde{f}_i J)_{\le x-1} \cup \{x+1\}.$$

Thus Proposition 7.8 implies

$$\begin{cases} \mathrm{base}(J_{\le x+1}) & = \mathrm{base}(\mathrm{base}(J_{\le x-1}) \cup \{x\}), \\ \mathrm{base}((\tilde{f}_i J)_{\le x+1}) & = \mathrm{base}(\mathrm{base}(J_{\le x-1}) \cup \{x+1\}). \end{cases}$$

As $\mathrm{base}(J_{\le x-1})$ is the set of beta numbers of an $e$-core, say $\mu$, and $s_i\mu \le \mu$ by (1), and the partition associated with $J_{\le x+1}$ has an addable $i$-node on the first row, we are in the situation of Lemma 8.1(4)(b). Note that the addable $i$-node on the first row is the lowest addable normal $i$-node. Thus, $(\tilde{f}_i J)_{\le x+1} = \tilde{f}_i\, J_{\le x+1}$ and

$$\mathrm{base}((\tilde{f}_i J)_{\le x+1}) = s_i \,\mathrm{base}(J_{\le x+1}) > \mathrm{base}(J_{\le x+1}).$$

(3) Denote $(\tilde{f}_i J)_{\le y-1} \cap \mathbb{Z}_{\ge x+2} = J_{\le y-1} \cap \mathbb{Z}_{\ge x+2}$ by $L$. $L$ does not contain beads on the $i^{th}$ and the $(i+1)^{th}$ runners. The former follows from the definition of $y$. To see the latter, observe that there is no bead between $x$ and $y$ on the $i^{th}$ runner. Thus, if there was a bead between $x+1$ and $y-e+1$ on the $(i+1)^{th}$ runner,

then $RA$-deletion would occur between $x$ and the bead, contradicting the fact that $x$ corresponds to a normal $i$-node. Hence the claim follows.

Write $L = \{j_s, \dots, j_{s+r}\}$ and set $J'_{s+r+1} = J_{\le x+1}$, $J''_{s+r+1} = (\tilde{f}_i J)_{\le x+1}$. Define $J'_k$ and $J''_k$, for $k = s+r, \dots, s$, by

$$J'_k = \mathrm{base}(J'_{k+1}) \cup \{j_k\} \text{ and } J''_k = \mathrm{base}(J''_{k+1}) \cup \{j_k\}.$$

We have

(♯) $\mathrm{base}(J''_{s+r+1}) = s_i \,\mathrm{base}(J'_{s+r+1}) > \mathrm{base}(J'_{s+r+1})$ by (2).

(♯) $\mathrm{base}(J_{\le y-1}) = \mathrm{base}(J'_s)$ and $\mathrm{base}((\tilde{f}_i J)_{\le y-1}) = \mathrm{base}(J''_s)$ by Corollary 7.9.

Suppose that $\mathrm{base}(J'_k) = \mathrm{base}(J''_k)$, for some $k$. Then we have

$$\mathrm{base}((\tilde{f}_i J)_{\le y-1}) = \mathrm{base}(J''_s) = \mathrm{base}(J'_s) = \mathrm{base}(J_{\le y-1}).$$

Next suppose that $\mathrm{base}(J'_k) \ne \mathrm{base}(J''_k)$, for all $k$. We prove by downward induction on $k$ that $\mathrm{base}(J''_k) = s_i \,\mathrm{base}(J'_k) > \mathrm{base}(J'_k)$. If $k = s+r+1$ then there is nothing to prove. Suppose that the assertion holds for $k+1$. Let

$$J'_{k+1,t} = \mathrm{down}^t(\mathrm{base}(J'_{k+1}) \cup \{j_k\}) \text{ and } J''_{k+1,t} = \mathrm{down}^t(\mathrm{base}(J''_{k+1}) \cup \{j_k\}),$$

for $t \ge 0$. We show that

$$(i)\ \ M_i(J'_{k+1,t}) > M_{i+1}(J'_{k+1,t}). \quad (ii)\ \ s_i J'_{k+1,t} = J''_{k+1,t}.$$

When $t = 0$ (i) and (ii) follow from $\mathrm{base}(J''_{k+1}) = s_i \,\mathrm{base}(J'_{k+1}) > \mathrm{base}(J'_{k+1})$.

Suppose (i) and (ii) for $t$ and apply the down operation to $J'_{k+1,t}$ and $J''_{k+1,t}$. Then, $p'$ is the same for both and it lies on the same runner as $j_k$. Consider $q'$ for $J'_{k+1,t}$. Then we have one of the following.

(a) If $q'$ is not on the $i^{th}$ or the $(i+1)^{th}$ runners, then $J'_{k+1,t+1}$ and $J''_{k+1,t+1}$ are obtained by moving $q'$ to $p' - e$ respectively.

(b) If $q'$ is on the $i^{th}$ runner, then $J'_{k+1,t+1}$ is obtained by moving $q'$ to $p' - e$ and $J''_{k+1,t+1}$ is obtained by moving $q' + 1$ to $p' - e$.

(c) If $q'$ is on the $(i+1)^{th}$ runner, then $J'_{k+1,t+1}$ is obtained by moving $q'$ to $p' - e$ and $J''_{k+1,t+1}$ is obtained by moving $q' - 1$ to $p' - e$.

In all the cases, we have (ii) for $t+1$. Now suppose that (i) breaks down at $t+1$. Then we have

$$M_i(J'_{k+1,t}) > M_{i+1}(J'_{k+1,t}) \text{ and } M_i(J'_{k+1,t+1}) \le M_{i+1}(J'_{k+1,t+1}).$$

The equality does not hold in the latter, since they are on different runners. Thus, we have $M_i(J'_{k+1,t+1}) = M_i(J'_{k+1,t}) - e$ and $M_{i+1}(J'_{k+1,t+1}) = M_{i+1}(J'_{k+1,t})$, and

$$M_i(J'_{k+1,t}) - e < M_{i+1}(J'_{k+1,t+1}) \le M_i(J'_{k+1,t}) - e + 1$$

implies that $M_{i+1}(J'_{k+1,t+1}) = M_i(J'_{k+1,t+1}) + 1$. Hence we conclude that $J''_{k+1,t+1} = s_i J'_{k+1,t+1} = J'_{k+1,t+1}$. However, this implies $\mathrm{base}(J'_k) = \mathrm{base}(J''_k)$, contradicting our assumption. Hence, (i) holds for $t+1$.

Therefore, $\mathrm{base}(J''_k) = s_i \,\mathrm{base}(J'_k) > \mathrm{base}(J'_k)$ holds. By setting $k = s$ and using $\mathrm{base}(J_{\le y-1}) = \mathrm{base}(J'_s)$ and $\mathrm{base}((\tilde{f}_i J)_{\le y-1}) = \mathrm{base}(J''_s)$, we have proved

$$\mathrm{base}((\tilde{f}_i J)_{\le y-1}) = s_i \,\mathrm{base}(J_{\le y-1}) > \mathrm{base}(J_{\le y-1})$$

in this case. □

**Lemma 8.3.** *Let $\lambda \in B(\Lambda_m)$ and $J$ the corresponding set of beta numbers of charge $m$. Suppose that $\tilde{f}_i \lambda \ne 0$ and $\tilde{f}_i J$ is obtained from $J$ by moving $x \in J$ to $x+1 \in \tilde{f}_i J$.*

(1) *Suppose that* $\{z \in J_{\geq x+1} \mid z + e\mathbb{Z} = i\} = \emptyset$.

  (a) *If* $s_i \operatorname{base}(\lambda) > \operatorname{base}(\lambda)$ *then* $\operatorname{base}(\tilde{f}_i\lambda) = s_i \operatorname{base}(\lambda) > \operatorname{base}(\lambda)$.

  (b) *If* $s_i \operatorname{base}(\lambda) \leq \operatorname{base}(\lambda)$ *then* $\operatorname{base}(\tilde{f}_i\lambda) = \operatorname{base}(\lambda)$.

(2) *If* $\{z \in J_{\geq x+1} \mid z + e\mathbb{Z} = i\} \neq \emptyset$ *then* $\operatorname{base}(\tilde{f}_i\lambda) = \operatorname{base}(\lambda)$.

*Proof.* (1) Write $J_{\geq x+2} = \{j_0, \ldots, j_r\}$. Set $J'_{r+1} = J_{\leq x+1}$ and $J''_{r+1} = (\tilde{f}_i J)_{\leq x+1}$. Then define $J'_k$ and $J''_k$, for $k = r, \ldots, 0$, by

$$J'_k = \operatorname{base}(J'_{k+1}) \cup \{j_k\} \ \text{ and } \ J''_k = \operatorname{base}(J''_{k+1}) \cup \{j_k\}.$$

Then Corollary 7.9 implies

$$\operatorname{base}(J) = \operatorname{base}(J'_0) \ \text{ and } \ \operatorname{base}(\tilde{f}_i J) = \operatorname{base}(J''_0).$$

There is no element of $J_{\geq x+2}$ on the $i^{th}$ runner because $M_i(J) = x$. Suppose that there is an element of $J_{\geq x+2}$ on the $(i+1)^{th}$ runner. We denote by $y$ the minimal such. Then $J$ has the following layers.

$$\begin{array}{ccc} \cdots & x & \cdots \\ \cdots & & \cdots \\ \cdots & \quad y & \cdots \end{array}$$

This implies that $RA$-deletion occurs between $x$ and $y$, which is a contradiction. Thus, there is also no element of $J_{\geq x+2}$ on the $(i+1)^{th}$ runner.

By Lemma 8.2(2), we have

$$\operatorname{base}(J''_{r+1}) = s_i \operatorname{base}(J'_{r+1}) > \operatorname{base}(J'_{r+1}).$$

Hence, $J''_r = s_i J'_r$ are $s_i$-cores and $M_i(J'_r) > M_{i+1}(J'_r)$.

We prove by downward induction on $k$ that

(a) If $s_i \operatorname{base}(J'_k) > \operatorname{base}(J'_k)$ then $\operatorname{base}(J''_k) = s_i \operatorname{base}(J'_k) > \operatorname{base}(J'_k)$.

(b) If $s_i \operatorname{base}(J'_k) \leq \operatorname{base}(J'_k)$ then $\operatorname{base}(J''_k) = \operatorname{base}(J'_k)$.

When $k = r$, (a) and (b) follow from Lemma 8.1(3). Suppose that (a) and (b) hold for $k+1$. Then we have either

(a') $J''_k = s_i J'_k$ are $s_i$-cores and $M_i(J'_k) > M_{i+1}(J'_k)$, or

(b') $J''_k = J'_k$ is an $s_i$-core and $M_i(J'_k) \leq M_{i+1}(J'_k)$.

Suppose that $s_i \operatorname{base}(J'_k) > \operatorname{base}(J'_k)$. Then (b') does not occur by Lemma 8.1(1). Thus (a') must occur and Lemma 8.1(3) implies

$$\operatorname{base}(J''_k) = s_i \operatorname{base}(J'_k) > \operatorname{base}(J'_k).$$

Suppose that $s_i \operatorname{base}(J'_k) \leq \operatorname{base}(J'_k)$. If (b') occurs then $\operatorname{base}(J''_k) = \operatorname{base}(J'_k)$ obviously holds, so we may assume that (a') occurs. Then, Lemma 8.1(3) implies $\operatorname{base}(J''_k) = \operatorname{base}(J'_k)$ also. We have proved that (a) and (b) hold for $k$.

Setting $k = 0$ and using $\operatorname{base}(J) = \operatorname{base}(J'_0)$ and $\operatorname{base}(\tilde{f}_i J) = \operatorname{base}(J''_0)$, we have the desired result.

(2) Define $y = \min\{z \in J_{\geq x+1} \mid z + e\mathbb{Z} = i\}$ as before. Then, by Proposition 7.8,

$$\begin{cases} \operatorname{base}(J_{\leq y}) & = \operatorname{base}(\operatorname{base}(J_{\leq y-1}) \cup \{y\}), \\ \operatorname{base}((\tilde{f}_i J)_{\leq y}) & = \operatorname{base}(\operatorname{base}((\tilde{f}_i J)_{\leq y-1}) \cup \{y\}). \end{cases}$$

Let $J' = \operatorname{base}(J_{\leq y-1}) \cup \{y\}$ and $J'' = \operatorname{base}((\tilde{f}_i J)_{\leq y-1}) \cup \{y\}$. By Lemma 8.2(3) we have either

(i) $\operatorname{base}((\tilde{f}_i J)_{\leq y-1}) = s_i \operatorname{base}(J_{\leq y-1}) > \operatorname{base}(J_{\leq y-1})$, or

(ii) $\mathrm{base}((\tilde{f}_i J)_{\le y-1}) = \mathrm{base}(J_{\le y-1})$.

Let $\lambda'$ be the partition whose set of beta numbers of charge $m$ is $J'$. If (i) occurs then $J'' = \tilde{f}_i^{\varphi_i(\lambda')-1} J'$ and we are in the situation of Lemma 8.1(4)(a). Thus we have $\mathrm{base}(J') = \mathrm{base}(J'')$. If (ii) occurs then $J' = J''$ and we have $\mathrm{base}(J') = \mathrm{base}(J'')$ again. Thus, $\mathrm{base}((\tilde{f}_i J)_{\le y}) = \mathrm{base}(J_{\le y})$ in both cases. Now Corollary 7.9 implies $\mathrm{base}(\tilde{f}_i J) = \mathrm{base}(J)$. □

The next theorem is the counterpart to Theorem 6.3, the "roof lemma" of [KLMW1].

**Theorem 8.4.** *Let $\lambda \in B(\Lambda_m)$. Then*

$$\mathrm{base}(\tilde{f}_i^{max}\lambda) = \begin{cases} s_i\,\mathrm{base}(\lambda) & (\textit{if } \mathrm{base}(\lambda) \textit{ has an addable } i\textit{-node}) \\ \mathrm{base}(\lambda) & (\textit{otherwise}) \end{cases}$$

*and* $\mathrm{base}(\tilde{f}_i^t\lambda) = \mathrm{base}(\lambda)$*, for* $0 \le t < \varphi_i(\lambda)$.

*Proof.* The theorem is equivalent to the following two statements.

(a) If $\varphi_i(\lambda) = 1$ and $s_i\,\mathrm{base}(\lambda) > \mathrm{base}(\lambda)$ then

$$\mathrm{base}(\tilde{f}_i\lambda) = s_i\,\mathrm{base}(\lambda) > \mathrm{base}(\lambda).$$

(b) Otherwise $\mathrm{base}(\tilde{f}_i\lambda) = \mathrm{base}(\lambda)$.

Suppose that the assumption in (a) holds. Then $s_i\,\mathrm{base}(\lambda) > \mathrm{base}(\lambda)$ implies $M_i(\lambda) > M_{i+1}(\lambda)$ by Lemma 8.1. Thus $M_i(\lambda)$ corresponds to an addable normal $i$-node. As $\varphi_i(\lambda) = 1$, $\tilde{f}_i\lambda$ is obtained from $\lambda$ by adding this node. We apply Lemma 8.3. Then $x = M_i(\lambda)$ and (1)(a) applies. Hence the result follows.

Suppose that the assumption in (b) holds. Then we have $s_i\,\mathrm{base}(\lambda) \le \mathrm{base}(\lambda)$ or $\varphi_i(\lambda) \ge 2$. In the former case, either (1)(b) or (2) of Lemma 8.3 applies. In the latter case, Lemma 8.3(2) applies. Hence $\mathrm{base}(\tilde{f}_i\lambda) = \mathrm{base}(\lambda)$ follows in both cases. □

**Corollary 8.5.** $\mathrm{base}(\lambda) = \mathrm{floor}(\lambda)$.

*Proof.* Note that Lemma 5.15 implies that

$$\mathrm{floor}(\tilde{f}_i^{max}\lambda) = \begin{cases} s_i\,\mathrm{floor}(\lambda) & (\text{if } \mathrm{floor}(\lambda) \text{ has an addable } i\text{-node}) \\ \mathrm{floor}(\lambda) & (\text{otherwise}) \end{cases}$$

and $\mathrm{floor}(\tilde{f}_i^t\lambda) = \mathrm{floor}(\lambda)$, for $0 \le t < \varphi_i(\lambda)$. Thus induction on the size of $\lambda$ proves the result. □

The next theorem follows from Theorem 6.23 and Corollary 8.5.

**Theorem 8.6.** *In the partition realization of $B(\Lambda_m)$, we have*

$$B^w(\Lambda_m) = \{\lambda \in B(\Lambda_m) \mid \mathrm{base}(\lambda) \supset w\emptyset_m\}.$$

Recall that $w_m$ is the longest element of $W_m$.

**Corollary 8.7.** *Write* $\mathrm{base}(\lambda) = w_\lambda\emptyset_m$*, for a unique $w_\lambda \in W/W_m$. Then*

$$w_\lambda w_m = \max\{w \in W \mid \lambda \in B^w(\Lambda_m)\}$$

*with respect to the Bruhat-Chevalley order.*

Recall that $M_i(\lambda)=\max\{x\in J \mid x+e\mathbb{Z}=i\}$.

**Definition 9.1.** *Let $\lambda\in B(\Lambda_0)$ be an e-core, J the corresponding set of beta numbers of charge 0. Write $\{M_i(\lambda)\}_{i\in\mathbb{Z}/e\mathbb{Z}}$ in descending order*

$$M_{i_1}(\lambda)>M_{i_2}(\lambda)>\cdots>M_{i_e}(\lambda).$$

*Then define $\tau_m(J)=J\cup\{M_{i_k}(\lambda)+e\}_{1\le k\le m}$, and denote the corresponding e-restricted partition by $\tau_m(\lambda)\in B(\Lambda_m)$. If $m=0$ then $\tau_m(\lambda)=\lambda$.*

Recall from the definition of $\mathring{W}$ in [Kc, p.74] and [Kc, Proposition 6.5] that $W$ is the semidirect product of $W_0$ and $T$, where $T=\{t_\alpha \mid \alpha\in\oplus_{i=1}^{e-1}\mathbb{Z}\alpha_i\}$, and $T$ acts on weights by

$$t_\alpha\Lambda=\Lambda+\Lambda(c)\alpha-((\Lambda,\alpha)+\frac{1}{2}|\alpha|^2\Lambda(c))\delta.$$

See [Kc, (6.5.2)]. Thus, any weight in the $W$-orbit $W\Lambda_0$ is of the form $t_\alpha\Lambda_0$, for some $t_\alpha\in T$. Note that $t_\alpha$ is not necessarily a distinguished coset representative.

**Lemma 9.2.** *Suppose that $\lambda\in B(\Lambda_0)$ is an e-core, and write $\lambda=t_\alpha\emptyset_0$, for $\alpha=\sum_{i=1}^{e-1}m_i\alpha_i$. Then $m_i=N_0(\lambda)-N_i(\lambda)$, for $1\le i\le e-1$.*

*Proof.* As $wt(\lambda)=t_\alpha\Lambda_0=\Lambda_0+\alpha-\frac{1}{2}|\alpha|^2\delta$,

$$\sum_{i=0}^{e-1}N_i(\lambda)\alpha_i=\Lambda_0-t_\alpha\Lambda_0=\frac{1}{2}|\alpha|^2\delta-\alpha.$$

Thus $N_0(\lambda)=\frac{1}{2}|\alpha|^2$ and $N_i(\lambda)=\frac{1}{2}|\alpha|^2-m_i$, for $1\le i\le e-1$. The result follows. ☐

**Proposition 9.3.** *Let $\lambda=w\emptyset_0\in B(\Lambda_0)$ and let $\mu=w'\emptyset_m\in B(\Lambda_m)$, where $w\in W/W_0$ and $w'\in W/W_m$. Then $ww_0\ge w'$ if and only if $\tau_m(\lambda)\supset\mu$.*

*Proof.* We use Proposition 4.4 throughout freely, without comment.

We may write $\lambda=t_\alpha\emptyset_0$, for $\alpha=\sum_{i=1}^{e-1}m_i\alpha_i$, and $t_\alpha=wv$, for $v\in W_0$. Then $ww_0\emptyset_m=t_\alpha u\emptyset_m$ for $u=v^{-1}w_0\in W_0$. If $u\in W_0$ then $t_\alpha u\le ww_0$, which implies $t_\alpha u\emptyset_m\subset ww_0\emptyset_m$. Thus

$$ww_0\emptyset_m=\max\{t_\alpha u\emptyset_m \mid u\in W_0\}.$$

If $ww_0\ge w'$ then $ww_0\emptyset_m\supset w'\emptyset_m=\mu$, and conversely, if $ww_0\emptyset_m\supset\mu$ then $ww_0\ge w'$. Thus we want to show $ww_0\emptyset_m=\tau_m(\lambda)$.

Suppose that $m=0$. Then $ww_0\emptyset_m=w\emptyset_m=\lambda$ and $ww_0\emptyset_m=\tau_m(\lambda)$ is trivial.

Suppose that $m\ne0$. Fix $u\in W_0$ and write $u\Lambda_m=\Lambda_m-\beta$, for some $\beta\in\sum_{i=1}^{e-1}\mathbb{Z}_{\ge0}\alpha_i$. Then

$$\begin{cases}t_\alpha\Lambda_m=\Lambda_m+\alpha-((\Lambda_m,\alpha)+\frac{1}{2}|\alpha|^2)\delta,\\ t_\alpha\beta=\beta-(\beta,\alpha)\delta.\end{cases}$$

We also have $t_\alpha\Lambda_0 = \Lambda_0 + \alpha - \frac{1}{2}|\alpha|^2\delta$, which implies $\sum_{i=0}^{e-1} N_i(\lambda)\alpha_i = \frac{1}{2}|\alpha|^2\delta - \alpha$ as before. Therefore,

$$\begin{aligned}\sum_{i=0}^{e-1} N_i(t_\alpha u\emptyset_m)\alpha_i = \Lambda_m - t_\alpha u\Lambda_m = \Lambda_m - t_\alpha(\Lambda_m - \beta)\\ = \big((\Lambda_m,\alpha) + \frac{1}{2}|\alpha|^2 - (\beta,\alpha)\big)\delta - \alpha + \beta\\ = (\Lambda_m - \beta,\alpha)\delta + \beta + \sum_{i=0}^{e-1} N_i(\lambda)\alpha_i.\end{aligned}$$

As $t_\alpha u\emptyset_m \subset ww_0\emptyset_m$, for all $u$, the height of $(\Lambda_m - \beta,\alpha)\delta + \beta$ must attain a maximum value at $ww_0\emptyset_m$.

As $u \in W_0$, we may compute $u\Lambda_m$ by restricting the weights to $\mathfrak{g}(A_{e-1})$. Hence we consider the restricted weights for the moment, and, by abuse of notation, we use the same $u\Lambda_m$. Then, $\Lambda_m$ may be considered as the weight $\epsilon_1 + \cdots + \epsilon_m$ of $\mathfrak{g}(A_{e-1}) = sl(e,\mathbb{C})$, where the weight lattice of $sl(e,\mathbb{C})$ is realized as $\oplus_{i=1}^{e-1}\mathbb{Z}\epsilon_i$ with $\sum_{i=1}^{e-1}\epsilon_i = 0$ as usual, and the simple roots are $\{\alpha_i = \epsilon_i - \epsilon_{i+1}\}_{1\le i<e}$. Thus,

$$u\Lambda_m = \Lambda_m - \beta \in \{\epsilon_{i_1} + \cdots + \epsilon_{i_m} \mid 1 \le i_1 < \cdots < i_m \le e\}.$$

Write $u\Lambda_m = \sum_{k=1}^m \epsilon_{i_k}$. Note that $(\epsilon_i,\epsilon_j) = \delta_{ij}$ and we may compute $(\Lambda_m - \beta,\alpha)$ by using the restricted weights. Thus, by Lemma 9.2,

$$(\Lambda_m - \beta,\alpha) = \sum_{k=1}^m (\epsilon_{i_k},\alpha) = \sum_{k=1}^m (m_{i_k} - m_{i_k-1}) = \sum_{k=1}^m (N_{i_k-1}(\lambda) - N_{i_k}(\lambda)).$$

As $\beta = \sum_{k=1}^m(\epsilon_k - \epsilon_{i_k})$, the height of $\beta$ is $\sum_{k=1}^m(i_k - k)$. Therefore, the value to be maximized is

$$\sum_{k=1}^m (N_{i_k-1}(\lambda) - N_{i_k}(\lambda))e + (i_k - k).$$

Define $L_i = (N_{i-1}(\lambda) - N_i(\lambda))e + i$, for $1 \le i \le e$. Here, we understand that $N_e(\lambda) = N_0(\lambda)$. It is important that the range for $i$ is not $0 \le i \le e-1$ but $1 \le i \le e$. Let $J$ be the set of beta numbers of charge 0 associated with $\lambda$ and $M_i(\lambda) = \max\{x \in J \mid x + e\mathbb{Z} = i\}$ as before. Then,

$$\sum_{i=0}^{e-1} N_i(t_\alpha u\emptyset_m)\alpha_i = (\sum_{k=1}^m \frac{L_{i_k} - i_k}{e})\delta + \sum_{k=1}^m (\alpha_k + \cdots + \alpha_{i_k-1}) + \sum_{i=0}^{e-1} N_i(\lambda)\alpha_i.$$

We claim that $L_i = M_i(\lambda) + e$, for $1 \le i \le e$. Recall how to read $N_i(\lambda)$ from the abacus. We explain this by an example. Let $\lambda = (4,2)$ and $e = 6$. Then the corresponding $J$ is displayed as follows.

$$\begin{array}{cccccc} -12 & -11 & -10 & -9 & -8 & -7 \\ -6 & -5 & -4 & -3 & -2 & \\ & 1 & & & 4 & \end{array}$$

We read the numbers on the abacus from $-\infty$ and with initial value 0, and increment the value by 1 at each number which does not belong to $J$. Equivalently, the value at $x$ is $|\{y \le x \mid y \notin J\}|$. We obtain

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
2 & 2 & 3 & 4 & 4 & 5 \\
6 & 7 & 8 & 9 & 10 & 11
\end{array}$$

We consider the same for the empty partition. Then we have

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 10 & 11
\end{array}$$

We compute the difference and obtain:

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Then $N_i(\lambda)$ is the summation of the entries on the $i^{th}$ runner.

$$N_0(\lambda) = 2,\ N_1(\lambda) = 1,\ N_2(\lambda) = 1,\ N_3(\lambda) = 1,\ N_4(\lambda) = 0,\ N_5(\lambda) = 1.$$

In this example, we have

$$L_1 = 7,\ L_2 = 2,\ L_3 = 3,\ L_4 = 10,\ L_5 = -1,\ L_6 = 0.$$

The proof of this rule is by induction on the size of $\lambda$. If $x \in J$ moves to $x+1$ when adding a node, then, as is explained in Example 2.1, the box to be added has the content $x$. Then observe that $|\{y \leq x \mid y \notin J\}|$ increases by 1 at $x$.

Let $\alpha$ and $\beta = \alpha + 1$ be two consecutive numbers such that $\alpha \in i - 1 + e\mathbb{Z}$ and $\beta \in i + e\mathbb{Z}$.

Suppose that $\alpha \geq 0$. Then, by the above rule for computing $N_i(\lambda)$, we have

(a) If $\beta \in J$ then the values at $\alpha$ and $\beta$ are the same. Thus, they contribute 1 to $N_{i-1}(\lambda) - N_i(\lambda)$.
(b) If $\beta \notin J$ then the value at $\beta$ is greater than the value at $\alpha$ by 1. Thus, they do not contribute to $N_{i-1}(\lambda) - N_i(\lambda)$.

Similarly, if $\alpha < 0$, then we have.

(a) If $\beta \notin J$ then they contribute $-1$ to $N_{i-1}(\lambda) - N_i(\lambda)$.
(b) If $\beta \in J$ then they do not contribute to $N_{i-1}(\lambda) - N_i(\lambda)$.

Suppose that $M_i(\lambda) \geq 1$. We have, for example,

$$\begin{array}{ccc}
\cdot & \times & \times \\
\cdot & & \times \\
0 & & \times \\
\cdot & & \times \\
\cdot & & \times
\end{array}$$

Then only those $\beta \in J$ with $\alpha \geq 0$ contribute and the number of such is $\frac{M_i(\lambda)+e-i}{e}$. Hence $L_i = \frac{M_i(\lambda)+e-i}{e}e + i = M_i(\lambda) + e$. Next suppose that $M_i(\lambda) \leq 0$.

$$\begin{array}{ccc} \cdot & \times & \times \\ \cdot & \times & \\ 0 & \times & \\ \cdot & \times & \\ \cdot & \times & \end{array}$$

Then only those $\beta \notin J$ with $\alpha < 0$ contribute and the number of such is $\frac{i-e-M_i(\lambda)}{e}$. Hence $L_i = -\frac{i-e-M_i(\lambda)}{e}e + i = M_i(\lambda) + e$. We have proved $L_i = M_i(\lambda) + e$.

Recall that we want to maximize $\sum_{k=1}^m L_{i_k}$. This is achieved precisely when $\{L_{i_k} - e \mid 1 \le k \le m\}$ consists of the largest $m$ numbers of $\{M_i(\lambda) \mid 1 \le i \le e\}$. ¿From now on, we suppose that

$$\{M_{i_1}(\lambda), M_{i_2}(\lambda), \dots, M_{i_m}(\lambda) \mid 1 \le i_1 < \dots < i_m \le e\}$$

are the largest $m$ numbers of $\{M_i(\lambda) \mid 1 \le i \le e\}$. We write $M_{i_k}$ for $M_{i_k}(\lambda)$. Then

$$\sum_{i=0}^{e-1} N_i(ww_0\emptyset_m)\alpha_i = (\sum_{k=1}^m \frac{M_{i_k} + e - i_k}{e})\delta + \sum_{k=1}^m (\alpha_k + \dots + \alpha_{i_k - 1}) + \sum_{i=0}^{e-1} N_i(\lambda)\alpha_i.$$

We compute $\Lambda_0 - wt(\lambda)$ and $\Lambda_m - wt(\tau_m(\lambda))$. For the computation, it is helpful to view a partition as a difference of two diagrams both of which extend infinitely to the left. Let $\mu \in B(\Lambda_m)$ and define two subsets of $\mathbb{Z}^2$ by

$$A = \{(i,j) \mid i \ge -m,\ j < \mu_{i+m}\} \text{ and } B = \{(i,j) \mid i \ge -m,\ j < 0\},$$

where the $i$-coordiate increases downward as in English convention. We also define the residue of $x = (i,j) \in \mathbb{Z}^2$ by $\mathrm{res}(x) = -i + j + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$. Then

$$\Lambda_m - wt(\mu) = \sum_{x \in A \setminus B} \alpha_{\mathrm{res}(x)} = \sum_{x \in A} \alpha_{\mathrm{res}(x)} - \sum_{x \in B} \alpha_{\mathrm{res}(x)}.$$

We can justify the rightmost by considering the region $D = \{(i,j) \mid i \le N, j \ge N'\}$, for sufficiently large $N$ and $-N'$, and understand it as

$$\sum_{x \in A \cap D} \alpha_{\mathrm{res}(x)} - \sum_{x \in B \cap D} \alpha_{\mathrm{res}(x)}.$$

Let $k_0 > k_1 > \cdots$ be the beta numbers of $\mu$. Thus, $k_j = \mu_j + m - j$. We may read them from $\mu$ as Example 2.1. Then we may write

$$\sum_{x \in A} \alpha_{\mathrm{res}(x)} = \sum_{j \ge 0} \sum_{s < k_j} \alpha_s, \text{ and } \sum_{x \in B} \alpha_{\mathrm{res}(x)} = \sum_{j \ge 0} \sum_{s < m-j} \alpha_s.$$

They do not make sense, but their difference does. Note that we can rearrange the order of a finite number of rows of $A$ or $B$ to compute $\Lambda_m - wt(\mu)$.

Now we compare $\Lambda_0 - wt(\lambda)$ and $\Lambda_m - wt(\tau_m(\lambda))$. Let $A = \{(i,j) \mid i \ge 0,\ j < \lambda_i\}$ and $B = \{(i,j) \mid i \ge 0,\ j < 0\}$. Then

$$\Lambda_0 - wt(\lambda) = \sum_{x \in A} \alpha_{\mathrm{res}(x)} - \sum_{x \in B} \alpha_{\mathrm{res}(x)}.$$

Define $A', B' \subset \mathbb{Z}^2$ by

$$A' = \{(-k,j) \mid 1 \le k \le m,\ j < M_{i_k} + e - k\}, \quad B' = \{(-k,j) \mid 1 \le k \le m,\ j < 0\}.$$

Then

$$\Lambda_m - wt(\tau_m(\lambda)) = \sum_{x\in A\cup A'} \alpha_{\mathrm{res}(x)} - \sum_{x\in B\cup B'} \alpha_{\mathrm{res}(x)}.$$

Thus $(\Lambda_m - wt(\tau_m(\lambda))) - (\Lambda_0 - wt(\lambda))$ is given by

$$\sum_{x\in A'} \alpha_{\mathrm{res}(x)} - \sum_{x\in B'} \alpha_{\mathrm{res}(x)}.$$

Observe that the first term is given by $\sum_{k=1}^m \sum_{j<M_{i_k}+e} \alpha_j$ and the second term is given by $\sum_{k=1}^m \sum_{j<k} \alpha_j$. Thus, for a sufficiently large $N$, we have

$$\Lambda_m - wt(\tau_m(\lambda)) = \sum_{k=1}^{m} \Big( \sum_{j=-N}^{M_{i_k}+e-1} \alpha_j - \sum_{j=-N}^{k-1} \alpha_j \Big) + \Lambda_0 - wt(\lambda),$$

and each term in the sum is equal to

$$\frac{M_{i_k}+e-i_k}{e}\delta + (\alpha_k + \cdots + \alpha_{i_k-1}).$$

Hence $\Lambda_m - wt(\tau_m(\lambda)) = \Lambda_m - wt(ww_0\emptyset_m)$, which implies $ww_0\emptyset_m = \tau_m(\lambda)$. □

We may describe $\tau_m(\lambda)$, for $1 \le m < e$, by Young diagrammatic terms. To see this, let $\ell = \ell(\lambda)$ be the length of $\lambda = (\lambda_0, \lambda_1, \dots)$ and define

$$\nu_i = \begin{cases} \lambda_i + e - m & (0 \le i < m) \\ \min\{\lambda_i + e - m, \lambda_{i-m}\} & (m \le i) \end{cases}$$

We have $\nu_i = 0$ if and only if $i \ge \ell + m$. It is clear that $\nu_0 \ge \cdots \ge \nu_{m-1}$ and $\nu_m \ge \nu_{m+1} \ge \cdots$. As $\nu_{m-1} < \nu_m$ would imply $\lambda_{m-1} + e - m < \lambda_m + e - m$, we have $\nu_{m-1} \ge \nu_m$. Hence, $\nu$ is a partition.

Let $\mathrm{shift}^m(\lambda) = (0^m, \lambda_0, \dots, \lambda_{\ell-1}, 0, \dots)$. We denote by $a^b$ the partition $(a^b, 0, \dots)$. The sum of partitions is defined by $\lambda + \mu = (\lambda_0 + \mu_0, \lambda_1 + \mu_1, \dots)$. The following proposition shows that

$$\tau_m(\lambda) = (\lambda + (e-m)^{\ell+m}) \cap ((\lambda_1 + e - m)^m + \mathrm{shift}^m(\lambda)).$$

In particular, we have

$$a(\tau_m(\lambda)) = a(\lambda) + e - m \text{ and } \ell(\tau_m(\lambda)) = \ell(\lambda) + m.$$

**Proposition 9.4.** *Let $\lambda$ be an $e$-core, and define $\nu$ as above. Then*

$$\nu = (\nu_0, \dots, \nu_{\ell+m-1}, 0, \dots) = \tau_m(\lambda).$$

*Proof.* Let $J$ be the set of beta numbers of charge 0 associated with $\lambda$, and let $K$ be the set of beta numbers of charge $m$ associated with $\nu$. Then

$$k_{i+m} = \min\{\lambda_{i+m} + e - m - i, \lambda_i - i\} = \min\{j_{i+m} + e, j_i\},$$

for $i \ge 0$. We also have $k_i = j_i + e$, for $0 \le i < m$. Hence, to obtain $K$ from $J$, we start with $J + e$, namely we slide down all the beads by one on the abacus, and move $j_{i+m} + e$ to $j_i$ when $j_{i+m} + e > j_i$, for $i \ge 0$. Since $\nu$ is a partition, $j_i = j_{i'+m} + e > j_{i'}$, for some $i'$, when it occurs.

Our aim is to prove that $K = J \cup \{M_{i_k}(\lambda) + e\}_{1\le k\le m}$. First we show that $x \in J$ implies $x \in K$. Suppose that $x = j_i$ and $x \notin K$. Since $x$ must move, $j_i = j_{i'+m} + e > j_{i'}$, for some $i' \ge 0$. Thus $i < i'$ and $j_{i+m} + e > j_{i'+m} + e = j_i$. Hence $j_{i+m} + e$ moves to $x$, which contradicts the assumption $x \notin K$.

Next consider $x \in \{M_i(\lambda)+e\}_{i\in\mathbb{Z}/e\mathbb{Z}}$. As $x \notin J$, no $j_{i'+m}+e \in J+e$ moves to $x$. Hence $x \notin K$ if and only if $x = j_{i+m}+e > j_i$, for some $i$. Let $x = j_{i+m}+e$. We have to show that $j_{i+m}+e > j_i$ if and only if $x \notin \{M_{i_k}(\lambda)+e\}_{1\le k\le m}$. If $j_{i+m}+e > j_i$ then $j_i > j_{i+1} > \cdots > j_{i+m-1} \ge j_{i+m}+1$ implies

$$\{j_{i+m-1}, j_{i+m-2}, \dots, j_i\} \subset \{j_{i+m}+1, j_{i+m}+2, \dots, j_{i+m}+e-1\} \cap J.$$

Hence $j_{i+m-1}+e, \dots, j_i+e$ are in pairwise distinct runners and all of them are greater than $x$. We have proved $x \notin \{M_{i_k}(\lambda)+e\}_{1\le k\le m}$. If $j_{i+m}+e \le j_i$ then there exists $i+m-1 \ge i' > i$ such that

$$\{j_{i+m-1}, j_{i+m-2}, \dots, j_{i'}\} = \{j_{i+m}+1, j_{i+m}+2, \dots, j_{i+m}+e-1\} \cap J.$$

In fact, it is clear that $j_{i+m-1}$ is the minimal element of the right hand side. Denote the maximal element by $j_{i'}$. Then $j_{i'} < j_{i+m}+e \le j_i$ implies $i' > i$.

These beads are in pairwise distinct runners. Each of the $i+m-i'(<m)$ runners has a bead which is greater than $x$, but the remaining runners do not have such a bead. Hence $x \in \{M_{i_k}(\lambda)+e\}_{1\le k\le m}$. □

We are now prepared to prove the following.

**Theorem 9.5.** *Let $\lambda \otimes \mu \in B(\Lambda_0) \otimes B(\Lambda_m)$. Then $\lambda \otimes \mu \in B(\Lambda_0+\Lambda_m)$ if and only if*

$$\tau_m(\mathrm{base}(\lambda)) \supset \mathrm{roof}(\mu).$$

*Proof.* Suppose that $m=0$. By Corollary 6.4 and Corollary 8.5, $\mathrm{base}(\lambda) \supset \mathrm{roof}(\mu)$ is equivalent to $\mathrm{floor}(\lambda) \supset \mathrm{ceil}(\mu)$. Write $\mathrm{floor}(\lambda) = w\emptyset_0$ and $\mathrm{base}(\mu) = w'\emptyset_0$, for $w, w' \in W/W_0$. Then $\mathrm{floor}(\lambda) \supset \mathrm{ceil}(\mu)$ is equivalent to $w \ge w'$, which is further equivalent to

$$f(\lambda) = w\Lambda_0 \ge w'\Lambda_0 = i(\mu).$$

Hence Corollary 5.8 for $r=d=2$ implies the result.

Suppose that $m \ne 0$. Write $\mathrm{base}(\lambda) = w\emptyset_0$ and $\mathrm{roof}(\mu) = w'\emptyset_m$, for $w \in W/W_0$ and $w' \in W/W_m$ respectively. Then Corollary 5.8 for $r=2, d=1$ implies that $\lambda \otimes \mu \in B(\Lambda_0+\Lambda_m)$ if and only if $ww_0 \ge w'$. This is equivalent to $\tau_m(\mathrm{base}(\lambda)) \supset \mathrm{roof}(\mu)$ by Proposition 9.3. □

Let $\mathcal{H}_n$ be the cyclotomic Hecke algebra defined by $(T_0+1)^d(T_0+q^m)^{r-d}=0$, $(T_i-q)(T_i+1)=0$, for $1 \le i < n$, and the type $B$ braid relations. As was mentioned in the introduction, a complete set of simple $\mathcal{H}_n$-modules is given by the set of nonzero $D^{(\lambda^{(r)},\dots,\lambda^{(1)})}$'s, where $D^{(\lambda^{(r)},\dots,\lambda^{(1)})}$ is obtained from the Specht module $S^{(\lambda^{(r)},\dots,\lambda^{(1)})}$ by factoring out the radical of the invariant symmetric bilinear form defined on it. The complete set is naturally a $\mathfrak{g}(A^{(1)}_{e-1})$-crystal $B(\Lambda)$, where $\Lambda = d\Lambda_0 + (r-d)\Lambda_m$. See [AM] and [A2], or [A1]. Note that when $r=2$ and $Q = -q^m$, we obtain the Hecke algebra $\mathcal{H}_n(Q,q)$ of type $B$ as special cases. Theorem 9.5 combined with the results explained in the introduction gives the following.

**Corollary 9.6.** *Let $\underline{\lambda} = \lambda^{(1)} \otimes \cdots \otimes \lambda^{(r)} \in B(\Lambda_0)^{\otimes d} \otimes B(\Lambda_m)^{\otimes r-d}$. Then the following are equivalent.*

(i) $D^{(\lambda^{(r)},\dots,\lambda^{(1)})} \ne 0$.
(ii) $\underline{\lambda} \in B(d\Lambda_0 + (r-d)\Lambda_m)$.
(iii) *The following three conditions hold.*
  (a) $\mathrm{base}(\lambda^{(k)}) \supset \mathrm{roof}(\lambda^{(k+1)})$, *for $1 \le k < d$,*
  (b) $\tau_m(\mathrm{base}(\lambda^{(d)})) \supset \mathrm{roof}(\lambda^{(d+1)})$,

(c) $\mathrm{base}(\lambda^{(k)}) \supset \mathrm{roof}(\lambda^{(k+1)})$, *for* $d < k < r$.

Recall that $a(\lambda)$ is the length of the first row, and $\ell(\lambda)$ is the length of the first column. For a multipartition $\underline{\lambda} = \lambda^{(1)} \otimes \cdots \otimes \lambda^{(r)}$, define $a_i(\underline{\lambda}) = a(\lambda^{(i)}) - \ell(\lambda^{(i+1)})$. Mathas proved the following result.

**Proposition 9.7.** *Suppose that* $e = 2$ *and let*

$$\underline{\lambda} = \lambda^{(1)} \otimes \cdots \otimes \lambda^{(r)} \in B(\Lambda_{m_1}) \otimes \cdots \otimes B(\Lambda_{m_r}) = B(\Lambda_0)^{\otimes d} \otimes B(\Lambda_m)^{\otimes r-d}.$$

*Then* $\underline{\lambda} \in B(d\Lambda_0 + (r-d)\Lambda_m)$ *if and only if* $a_i(\underline{\lambda}) \geq \delta_{m_i m_{i+1}} - 1$, *for* $1 \leq i < r$.

Observe that any 2-core $\lambda$ is of the form $(c, c-1, \ldots, 1)$ and $a(\lambda) = \ell(\lambda) = c$. Using the closed formulas for $\mathrm{ceil}(\lambda)$ and $\mathrm{floor}(\lambda)$ for a partition $\lambda$ which is given in Proposition 5.22, we have

(i) If $m_i = m_{i+1}$ then $\mathrm{floor}(\lambda^{(i)}) \supset \mathrm{ceil}(\lambda^{(i+1)})$ is equivalent to

$$a(\lambda^{(i)}) \geq \ell(\lambda^{(i+1)}).$$

(ii) If $m_i \neq m_{i+1}$ then $\tau_1(\mathrm{floor}(\lambda^{(i)})) \supset \mathrm{ceil}(\lambda^{(i+1)})$ is equivalent to

$$a(\lambda^{(i)}) + 1 \geq \ell(\lambda^{(i+1)}).$$

Thus, Mathas' result follows from our results.

Now consider $e = 3$. Recently, in the spirit similar to Mathas' result in $e = 2$, Fayers has obtained a necessary and sufficient condition for $(\lambda, \mu)$ to be a Kleshchev bipartition [F]. According to him, the condition may be restated as follows.

**Proposition 9.8.** *Suppose that* $e = 3$ *and let* $\lambda \otimes \mu \in B(\Lambda_0) \otimes B(\Lambda_m)$.

(i) *If* $m = 0$ *then* $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m)$ *if and only if*

$$a(\lambda) \geq \ell(m(\mu)) \text{ and } a(m(\lambda)) \geq \ell(\mu).$$

(ii) *If* $m = 1$ *then* $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m)$ *if and only if*

$$a(\lambda) \geq \ell(m(\mu)) - 2 \text{ and } a(m(\lambda)) \geq \ell(\mu) - 1.$$

(iii) *If* $m = 2$ *then* $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m)$ *if and only if*

$$a(\lambda) \geq \ell(m(\mu)) - 1 \text{ and } a(m(\lambda)) \geq \ell(\mu) - 2.$$

Recall that $\ell(\mathrm{roof}(\mu)) = \ell(\mu)$ by Lemma 2.4(3), and $a(\mathrm{base}(\lambda)) = a(\lambda)$ by Lemma 2.7(3). By Proposition 5.21, we have the following equalities.

(i) $a(\lambda) = a(\mathrm{base}(\lambda))$ and $\ell(m(\mu)) = \ell(\mathrm{roof}(m(\mu))) = a(\mathrm{roof}(\mu))$.

(ii) $a(m(\lambda)) = a(\mathrm{base}(m(\lambda))) = \ell(\mathrm{base}(\lambda))$ and $\ell(\mu) = \ell(\mathrm{roof}(\mu))$.

Thus, his condition is precisely

$$a(\tau_m(\mathrm{base}(\lambda))) \geq a(\mathrm{roof}(\mu)) \text{ and } \ell(\tau_m(\mathrm{base}(\lambda))) \geq \ell(\mathrm{roof}(\mu)).$$

Note that any 3-core $\lambda$ is of the form $(c, c-2, \ldots, c-2r+2, d^2, (d-1)^2, \ldots, 1^2)$, where $d = c - 2r$ or $d = c - 2r + 1$.[7] In particular, $\lambda$ is determined by $a(\lambda)$ and $\ell(\lambda)$, because $a(\lambda) = c$ and $\ell(\lambda) = r + 2d = 2c - 3r$ or $2c - 3r + 2$ imply

$$r = -[\frac{\ell(\lambda) - 2a(\lambda)}{3}], \quad d = [\frac{2\ell(\lambda) - a(\lambda)}{3}].$$

Hence, the above condition is equivalent to $\tau_m(\mathrm{base}(\lambda)) \supset \mathrm{roof}(\mu)$.

[7] The number of $i$ such that $\lambda_i = \lambda_{i+1} + 2$ is $r$ in the former case, and $r - 1$ in the latter case.

As a conclusion, we may deduce Proposition 9.8 from our results, and conversely, we may restate our results Theorem 9.5 and Corollary 9.6 in $e = 3$ by using his more explicit numerical conditions, which we do not mention here.

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