RICHARDSON VARIETIES IN THE GRASSMANNIAN

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Dedicated to Professor J. Shalika on his sixtieth birthday

Abstract. The Richardson variety X_w^v is defined to be the intersection of the Schubert variety X_w and the opposite Schubert variety X^v. For X_w^v in the Grassmannian, we obtain a standard monomial basis for the homogeneous coordinate ring of X_w^v. We use this basis first to prove the vanishing of $H^i(X_w^v, L^m)$, $i > 0$, $m \geq 0$, where L is the restriction to X_w^v of the ample generator of the Picard group of the Grassmannian; then to determine a basis for the tangent space and a criterion for smoothness for X_w^v at any T-fixed point e_τ; and finally to derive a recursive formula for the multiplicity of X_w^v at any T-fixed point e_τ. Using the recursive formula, we show that the multiplicity of X_w^v at e_τ is the product of the multiplicity of X_w at e_τ and the multiplicity of X^v at e_τ. This result allows us to generalize the Rosenthal-Zelevinsky determinantal formula for multiplicities at T-fixed points of Schubert varieties to the case of Richardson varieties.

Contents

Introduction

Let G denote a semisimple, simply connected, algebraic group defined over an algebraically closed field K of arbitrary characteristic. Let us fix a maximal torus T and a Borel subgroup B containing T. Let W be the Weyl group ($N(T)/T$, $N(T)$ being the normalizer of T). Let Q be a parabolic subgroup of G containing B, and W_Q, the

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Weyl group of Q. For the action of G on G/Q given by left multiplication, the T-fixed points are precisely the cosets $e_w := wQ$ in G/Q. For $w \in W/W_Q$, let X_w denote the *Schubert variety* (the Zariski closure of the B-orbit Be_w in G/Q through the T-fixed point e_w), endowed with the canonical structure of a closed, reduced subscheme of G/Q. Let B^- denote the Borel subgroup of G opposite to B (it is the unique Borel subgroup of G with the property $B \cap B^- = T$). For $v \in W/W_Q$, let X^v denote the *opposite Schubert variety*, the Zariski closure of the B^--orbit B^-e_v in G/Q.

Schubert and opposite Schubert varieties play an important role in the study of the generalized flag variety G/Q, especially, the algebraic-geometric and representation-theoretic aspects of G/Q. A more general class of subvarieties in G/Q is the class of Richardson varieties; these are varieties of the form $X_w^v := X_w \cap X^v$, the intersection of the Schubert variety X_w with opposite Schubert variety X^v. Such varieties were first considered by Richardson in (cf. [22]), who shows that such intersections are reduced and irreducible. Recently, Richardson varieties have shown up in several contexts: such double coset intersections $BwB \cap B^-xB$ first appear in [11], [12], [22], [23]. Very recently, Richardson varieties have also appeared in the context of K-theory of flag varieties ([3], [14]). They also show up in the construction of certain degenerations of Schubert varieties (cf. [3]).

In this paper, we present results for Richardson varieties in the Grassmannian variety. Let $G_{d,n}$ be the Grassmannian variety of d-dimensional subspaces of K^n, and $p : G_{d,n} \hookrightarrow \mathbb{P}^N (= \mathbb{P}(\wedge^d K^n))$, the Plücker embedding (note that $G_{d,n}$ may be identified with $G/P, G = SL_n(K), P$ a suitable maximal parabolic subgroup of G). Let $X := X_w \cap X^v$ be a Richardson variety in $G_{d,n}$. We first present a Standard monomial theory for X (cf. Theorem 3.3.2). Standard monomial theory (SMT) consists in constructing an explicit basis for the homogeneous coordinate ring of X. SMT for Schubert varieties was first developed by the second author together with Musili and Seshadri in a series of papers, culminating in [16], where it is established for all classical groups. Further results concerning certain exceptional and Kac–Moody groups led to conjectural formulations of a general SMT, see [17]. These conjectures were then proved by Littelmann, who introduced new combinatorial and algebraic tools: the path model of representations of any Kac-Moody group, and Lusztig's Frobenius map for quantum groups at roots of unity (see [18, 19]); recently, in collaboration with Littelmann (cf. [14]), the second author has extended the results of [19] to Richardson varieties in G/B, for any semisimple G. Further, in collaboration with Brion (cf. [4]), the second author has also given a purely geometric construction of standard monomial basis for Richardson varieties in G/B, for any semisimple G; this construction in loc. cit. is done using certain flat family with generic fiber $\cong \mathrm{diag}(X_w^v) \subset X_w^v \times X_w^v$, and the special fiber $\cong \cup_{v \le x \le w} X_x^v \times X_w^x$.

If one is concerned with just Richardson varieties in the Grassmannian, one could develop a SMT in the same spirit as in [21] using just the Plücker coordinates, and one doesn't need to use any quantum group theory nor does one need the technicalities of [4]. Thus we give a self-contained presentation of SMT for unions of Richardson varieties in the Grassmannian. We should remark that Richardson vatieties in the Grassmannian are also studied in [26], where these varieties are called *skew Schubert varieties*, and standard monomial bases for these varieties also appear in loc. cit. (Some discussion of these varieties also appears in [10].) As a consequence of our results for unions of Richardson varieties, we deduce the vanishing of $H^i(X, L^m), i \geq 1, m \geq 0, L$, being the restriction to X of $\mathcal{O}_{\mathbb{P}^N}(1)$ (cf. Theorem 5.0.6); again, this result may be deduced using the theory of Frobenius-splitting (cf. [20]), while our approach uses just the classical Pieri formula. Using the standard monomial basis, we then determine the tangent space and also the multiplicity at any T-fixed point e_τ on X. We first give a recursive formula for the multiplicity of X at e_τ (cf. Theorem 7.6.2). Using the recursive formula, we derive a formula for the multiplicity of X at e_τ as being the product of the multiplicities at e_τ of X_w and X^v (as above, $X = X_w \cap X^v$) (cf. Theorem 7.6.4). Using the product formula, we get a generalization of Rosenthal-Zelevinsky determinantal formula (cf. [24]) for the multiplicities at singular points of Schubert varieties to the case of Richardson varieties (cf. Theorem 7.7.3). It should be mentioned that the multiplicities of Schubert varieties at T-fixed points determine their multiplicities at all other points, because of the B-action; but this does not extend to Richardson varieties, since Richardson varieties have only a T-action. Thus even though, certain smoothness criteria at T-fixed points on a Richardson variety are given in Corollaries 6.7.3 and 7.6.5, the problem of the determination of singular loci of Richardson varieties still remains open.

In §1, we present basic generalities on the Grassmannian variety and the Plücker embedding. In §2, we define Schubert varieties, opposite Schubert varieties, and the more general Richardson varieties in the Grassmannian and give some of their basic properties. We then develop a standard monomial theory for a Richardson variety X_w^v in the Grassmannian in §3 and extend this to a standard monomial theory for unions and nonempty intersections of Richardson varieties in the Grassmannian in §4. Using the standard monomial theory, we obtain our main results in the three subsequent sections. In §5, we prove the vanishing of $H^i(X_w^v, L^m)$, $i > 0$, $m \geq 0$, where L is the restriction to X_w^v of the ample generator of the Picard group of the Grassmannian. In §6, we determine a basis for the tangent space and a criterion for smoothness for X_w^v at any T-fixed point e_τ. Finally, in §7, we derive several formulas for the multiplicity of X_w^v at any T-fixed point e_τ.

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1. The Grassmannian Variety $G_{d,n}$

Let K be the base field, which we assume to be algebraically closed of arbitrary characteristic. Let d be such that $1 \leq d < n$. The *Grassmannian* $G_{d,n}$ is the set of all

d-dimensional subspaces of K^n. Let U be an element of $G_{d,n}$ and $\{a_1, \dots a_d\}$ a basis of U, where each a_j is a vector of the form

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}, \text{ with } a_{ij} \in K.$$

Thus, the basis $\{a_1, \cdots, a_d\}$ gives rise to an $n \times d$ matrix $A = (a_{ij})$ of rank d, whose columns are the vectors $a_1, \cdots, a_d$.

We have a canonical embedding

$$p : G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d K^n)\ ,\ U \mapsto [a_1 \wedge \cdots \wedge a_d]$$

called the *Plücker embedding*. It is well known that p is a closed immersion; thus $G_{d,n}$ acquires the structure of a projective variety. Let

$$I_{d,n} = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{N}^d\ :\ 1 \leq i_1 < \cdots < i_d \leq n\}.$$

Then the projective coordinates (*Plücker coordinates*) of points in $\mathbb{P}(\wedge^d K^n)$ may be indexed by $I_{d,n}$; for $\underline{i} \in I_{d,n}$, we shall denote the $\underline{i}$-th component of p by $p_{\underline{i}}$, or $p_{i_1,\cdots,i_d}$. If a point U in $G_{d,n}$ is represented by the $n \times d$ matrix A as above, then $p_{i_1,\cdots,i_d}(U) = \det(A_{i_1,\dots,i_d})$, where $A_{i_1,\dots,i_d}$ denotes the $d \times d$ submatrix whose rows are the rows of A with indices $i_1, \dots, i_d$, in this order.

For $\underline{i} \in I_{d,n}$ consider the point $e_{\underline{i}}$ of $G_{d,n}$ represented by the $n \times d$ matrix whose entries are all 0, except the ones in the i_j-th row and j-th column, for each $1 \leq j \leq d$, which are equal to 1. Clearly, for $\underline{i}, \underline{j} \in I_{d,n}$,

$$p_{\underline{i}}(e_{\underline{j}}) = \begin{cases} 1, & \text{if } \underline{i} = \underline{j}; \\ 0, & \text{otherwise.} \end{cases}$$

We define a partial order $\geq$ on $I_{d,n}$ in the following manner: if $\underline{i} = (i_1, \cdots, i_d)$ and $\underline{j} = (j_1, \cdots, j_d)$, then $\underline{i} \geq \underline{j} \Leftrightarrow i_t \geq j_t, \forall t$. The following well known theorem gives the defining relations of $G_{d,n}$ as a closed subvariety of $\mathbb{P}(\wedge^d K^n)$ (cf. [9]; see [13] for details):

Theorem 1.0.1. *The Grassmannian $G_{d,n} \subset \mathbb{P}(\wedge^d K^n)$ consists of the zeroes in $\mathbb{P}(\wedge^d K^n)$ of quadratic polynomials of the form*

$$p_{\underline{i}} p_{\underline{j}} - \sum \pm p_\alpha p_\beta$$

for all $\underline{i}$, $\underline{j} \in I_{d,n}$, $\underline{i}$, $\underline{j}$ non-comparable, where α, β run over a certain subset of $I_{d,n}$ such that $\alpha >$ both $\underline{i}$ and $\underline{j}$, and $\beta <$ both $\underline{i}$ and $\underline{j}$.

1.1. **Identification of G/P_d with $G_{d,n}$.** Let $G = SL_n(K)$. Let P_d be the maximal parabolic subgroup

$$P_d = \left\{ A \in G \,\middle|\, A = \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \right\}.$$

For the natural action of G on $\mathbb{P}(\wedge^d K^n)$, we have, the isotropy at $[e_1 \wedge \cdots \wedge e_d]$ is P_d while the orbit through $[e_1 \wedge \cdots \wedge e_d]$ is $G_{d,n}$. Thus we obtain a surjective morphism $\pi : G \to G_{d,n},\ g \mapsto g \cdot a$, where $a = [e_1 \wedge \cdots \wedge e_d]$. Further, the differential $d\pi_e$: Lie$G \to T(G_{d,n})_a$ (= the tangent space to $G_{d,n}$ at a) is easily seen to be surjective. Hence we obtain an identification $f_d : G/P_d \cong G_{d,n}$ (cf. [1], Proposition 6.7).

1.2. **Weyl Group and Root System.** Let G and P_d be as above. Let T be the subgroup of diagonal matrices in G, B the subgroup of upper triangular matrices in G, and B^- the subgroup of lower triangular matrices in G. Let W be the Weyl group of G relative to T, and W_{P_d} the Weyl group of P_d. Note that $W = S_n$, the group of permutations of a set of n elements, and that $W_{P_d} = S_d \times S_{n-d}$. For a permutation w in S_n, $l(w)$ will denote the usual length function. Note also that $I_{d,n}$ can be identified with W/W_{P_d}. In the sequel, we shall identify $I_{d,n}$ with the set of "minimal representatives" of W/W_{P_d} in S_n; to be very precise, a d-tuple $\underline{i} \in I_{d,n}$ will be identified with the element $(i_1, \dots, i_d, j_1, \dots, j_{n-d}) \in \mathcal{S}_n$, where $\{j_1, \dots, j_{n-d}\}$ is the complement of $\{i_1, \dots, i_d\}$ in $\{1, \dots, n\}$ arranged in increasing order. We denote the set of such minimal representatives of S_n by W^{P_d}.

Let R denote the root system of G relative to T, and R^+ the set of positive roots relative to B. Let R_{P_d} denote the root system of P_d, and $R^+_{P_d}$ the set of positive roots.

2. Schubert, Opposite Schubert, and Richardson Varieties in $G_{d,n}$

For $1 \le t \le n$, let V_t be the subspace of K^n spanned by $\{e_1, \dots, e_t\}$, and let V^t be the subspace spanned by $\{e_n, \dots, e_{n-t+1}\}$. For each $\underline{i} \in I_{d,n}$, the *Schubert variety* $X_{\underline{i}}$ and *Opposite Schubert variety* $X^{\underline{i}}$ associated to $\underline{i}$ are defined to be

$$X_{\underline{i}} = \{U \in G_{d,n} \mid \dim(U \cap V_{i_t}) \ge t\ ,\ 1 \le t \le d\},$$

$$X^{\underline{i}} = \{U \in G_{d,n} \mid \dim(U \cap V^{n-i_{(d-t+1)}+1}) \ge t\ ,\ 1 \le t \le d\}.$$

For $\underline{i}, \underline{j} \in I_{d,n}$, the *Richardson Variety* $X^{\underline{j}}_{\underline{i}}$ is defined to be $X_{\underline{i}} \cap X^{\underline{j}}$. For $\underline{i}, \underline{j}, \underline{e}, \underline{f} \in I_{d,n}$, where $\underline{e} = (1, \dots, d)$ and $\underline{f} = (n+1-d, \dots, n)$, note that $G_{d,n} = X^{\underline{e}}_{\underline{f}}$, $X_{\underline{i}} = X^{\underline{e}}_{\underline{i}}$, and $X^{\underline{i}} = X^{\underline{i}}_{\underline{f}}$.

For the action of G on $\mathbb{P}(\wedge^d K^n)$, the T-fixed points are precisely the points corresponding to the T-eigenvectors in $\wedge^d K^n$. Now

$$\wedge^d K^n = \bigoplus_{\underline{i} \in I_{d,n}} K e_{\underline{i}}, \quad \text{as } T\text{-modules},$$

where for $\underline{i} = (i_1, \cdots, i_d)$, $e_{\underline{i}} = e_{i_1} \wedge \cdots \wedge e_{i_d}$. Thus the T-fixed points in $\mathbb{P}(\wedge^d K^n)$ are precisely $[e_{\underline{i}}]$, $\underline{i} \in I_{d,n}$, and these points, obviously, belong to $G_{d,n}$. Further, the Schubert variety $X_{\underline{i}}$ associated to $\underline{i}$ is simply the Zariski closure of the B-orbit $B\,[e_{\underline{i}}]$ through the T-fixed point $[e_{\underline{i}}]$ (with the canonical reduced structure), B being as in §1.2. The opposite Schubert variety $X^{\underline{i}}$ is the Zariski closure of the B^--orbit $B^-\,[e_{\underline{i}}]$ through the T-fixed point $[e_{\underline{i}}]$ (with the canonical reduced structure), B^- being as in §1.2.

2.1. **Bruhat Decomposition.** Let $V = K^n$. Let $\underline{i} \in I_{d,n}$. Let $C_{\underline{i}} = B[e_{\underline{i}}]$ be the *Schubert cell* and $C^{\underline{i}} = B^-[e_{\underline{i}}]$ the *opposite Schubert cell* associated to $\underline{i}$. The $C_{\underline{i}}$'s provide a cell decomposition of $G_{d,n}$, as do the $C^{\underline{i}}$'s. Let $X = V \oplus \cdots \oplus V$ (d times). Let

$$\pi : X \to \wedge^d V, (u_1, \ldots, u_d) \mapsto u_1 \wedge \ldots \wedge u_d,$$

and

$$p : \wedge^d V \setminus \{0\} \to \mathbb{P}(\wedge^d V), u_1 \wedge \ldots \wedge u_d \mapsto [u_1 \wedge \ldots \wedge u_d].$$

Let $v_{\underline{i}}$ denote the point $(e_{i_1}, \ldots, e_{i_d}) \in X$.

Identifying X with $M_{n\times d}$, $v_{\underline{i}}$ gets identified with the $n \times d$ matrix whose entries are all zero except the ones in the i_j-th row and j-th column, $1 \leq j \leq d$, which are equal to 1. We have

$$B \cdot v_{\underline{i}} = \{A \in M_{n\times d} \mid x_{ij} = 0, i > i_j, \text{ and } \prod_t x_{i_t t} \neq 0\},$$

$$B^- \cdot v_{\underline{i}} = \{A \in M_{n\times d} \mid x_{ij} = 0, i < i_j, \text{ and } \prod_t x_{i_t t} \neq 0\}.$$

Denoting $\overline{B \cdot v_{\underline{i}}}$ by $D_{\underline{i}}$, we have $D_{\underline{i}} = \{A \in M_{n\times d} \mid x_{ij} = 0, i > i_j\}$. Further, $\pi(B \cdot v_{\underline{i}}) = p^{-1}(C_{\underline{i}})$, $\pi(D_{\underline{i}}) = \widehat{X_{\underline{i}}}$, the cone over $X_{\underline{i}}$. Denoting $\overline{B^- \cdot v_{\underline{i}}}$ by $D^{\underline{i}}$, we have $D^{\underline{i}} = \{A \in M_{n\times d} \mid x_{ij} = 0, i < i_j\}$. Further, $\pi(B^- \cdot v_{\underline{i}}) = p^{-1}(C^{\underline{i}})$, $\pi(D^{\underline{i}}) = \widehat{X^{\underline{i}}}$, the cone over $X^{\underline{i}}$. From this, we obtain

Theorem 2.1.1. (1) *Bruhat Decomposition:* $X_{\underline{j}} = \dot{\bigcup}_{\underline{i} \leq \underline{j}} Be_{\underline{i}}$, $X^{\underline{j}} = \dot{\bigcup}_{\underline{i} \geq \underline{j}} B^- e_{\underline{i}}$.

(2) $X_{\underline{i}} \subseteq X_{\underline{j}}$ *if and only if* $\underline{i} \leq \underline{j}$.
(3) $X^{\underline{i}} \subseteq X^{\underline{j}}$ *if and only if* $\underline{i} \geq \underline{j}$.

Corollary 2.1.2. (1) $X_{\underline{j}}^{\underline{k}}$ *is nonempty* $\iff$ $\underline{j} \geq \underline{k}$*; further, when* $X_{\underline{j}}^{\underline{k}}$ *is nonempty, it is reduced and irreducible of dimension* $l(w) - l(v)$*, where* w *(resp.* v*) is the permutation in* S_n *representing* $\underline{j}$ *(resp.* $\underline{k}$*) as in §1.2.*
(2) $p_{\underline{j}}\big|_{X_{\underline{i}}^{\underline{k}}} \neq 0 \iff \underline{i} \geq \underline{j} \geq \underline{k}$.

Proof. (1) Follows from [22]. The criterion for $X_{\underline{j}}^{\underline{k}}$ to be nonempty, the irreducibility, and the dimension formula are also proved in [6].
(2) From Bruhat decomposition, we have $p_{\underline{j}}|_{X_{\underline{i}}} \neq 0 \iff e_{\underline{j}} \in X_{\underline{i}}$; we also have $p_{\underline{j}}|_{X^{\underline{k}}} \neq 0 \iff e_{\underline{j}} \in X^{\underline{k}}$. Thus $p_{\underline{j}}|_{X_{\underline{i}}^{\underline{k}}} \neq 0 \iff e_{\underline{j}} \in X_{\underline{i}}^{\underline{k}}$. Again from Bruhat decomposition, we have $e_{\underline{j}} \in X_{\underline{i}}^{\underline{k}} \iff \underline{i} \geq \underline{j} \geq \underline{k}$. The result follows from this. □

For the remainder of this paper, we will assume that all our Richardson varieties are nonempty.

Remark 2.1.3. In view of Theorem 2.1.1, we have $X_{\underline{i}} \subseteq X_{\underline{j}}$ if and only if $\underline{i} \leq \underline{j}$. Thus, under the set-theoretic bijection between the set of Schubert varieties and the set $I_{d,n}$, the partial order on the set of Schubert varieties given by inclusion induces the partial order $\geq$ on $I_{d,n}$.

2.2. **More Results on Richardson Varieties.**

Lemma 2.2.1. *Let $X \subseteq G_{d,n}$ be closed and B-stable (resp. B^--stable). Then X is a union of Schubert varieties (resp. opposite Schubert varieties).*

The proof is obvious.

Lemma 2.2.2. *Let X_1, X_2 be two Richardson varieties in $G_{d,n}$ with nonempty intersection. Then $X_1 \cap X_2$ is a Richardson variety (set-theoretically).*

Proof. We first give the proof when X_1 and X_2 are both Schubert varieties. Let $X_1 = X_{\tau_1}$, $X_2 = X_{\tau_2}$, where $\tau_1 = (a_1, \ldots, a_d)$, $\tau_2 = (b_1, \ldots, b_d)$. By Lemma 2.2.1, $X_1 \cap X_2 = \cup X_{w_i}$, where $w_i < \tau_1$, $w_i < \tau_2$. Let $c_j = \min\{a_j, b_j\}$, $1 \leq j \leq d$, and $\tau = (c_1, \ldots, c_d)$. Then, clearly $\tau \in I_{d,n}$, and $\tau < \tau_i$, $i = 1, 2$. We have $w_i \leq \tau$, and hence $X_1 \cap X_2 = X_\tau$.

The proof when X_1 and X_2 are opposite Schubert varieties is similar. The result for Richardson varieties follows immediately from the result for Schubert varieties and the result for opposite Schubert varieties. □

Remark 2.2.3. Explicitly, in terms of the distributive lattice structure of $I_{d,n}$, we have that $X_{w_1}^{v_1} \cap X_{w_2}^{v_2} = X_{w_1 \wedge w_2}^{v_1 \vee v_2}$ (set theoretically), where $w_1 \wedge w_2$ is the *meet* of w_1 and w_2 (the largest element of W^{P_d} which is less than both w_1 and w_2) and $v_1 \vee v_2$ is the *join* of v_1 and v_2 (the smallest element of W^{P_d} which is greater than both v_1 and v_2). The fact that $X_1 \cap X_2$ is reduced follows from [20]; we will also provide a proof in Theorem 4.3.1.

3. Standard Monomial Theory for Richardson Varieties

3.1. **Standard Monomials.**

Let R_0 be the homogeneous coordinate ring of $G_{d,n}$ for the Plücker embedding, and for $w, v \in I_{d,n}$, let R_w^v be the homogeneous coordinate ring of the Richardson variety X_w^v. In this section, we present a standard monomial theory for X_w^v in the same spirit as in [21]. As mentioned in the introduction, standard monomial theory consists in constructing an explicit basis for R_w^v.

Definition 3.1.1. A monomial $f = p_{\tau_1} \cdots p_{\tau_m}$ is said to be *standard* if

$$(*) \qquad \tau_1 \geq \cdots \geq \tau_m.$$

Such a monomial is said to be *standard on* X_w^v, if in addition to condition (*), we have $w \geq \tau_1$ and $\tau_m \geq v$.

Remark 3.1.2. Note that in the presence of condition (*), the standardness of f on X_w^v is equivalent to the condition that $f\big|_{X_w^v} \neq 0$. Thus given a standard monomial f, we have $f\big|_{X_w^v}$ is either 0 or remains standard on X_w^v.

3.2. **Linear Independence of Standard Monomials.**

Theorem 3.2.1. *The standard monomials on X_w^v of degree m are linearly independent in R_w^v.*

Proof. We proceed by induction on $\dim X_w^v$.

If $\dim X_w^v = 0$, then $w = v$, p_w^m is the only standard monomial on X_w^v of degree m, and the result is obvious. Let $\dim X_w^v > 0$. Let

$$(*) \qquad 0 = \sum_{i=1}^{r} c_i F_i,\ c_i \in K^*\,,$$

be a linear relation of standard monomials F_i of degree m. Let $F_i = p_{w_{i1}} \ldots p_{w_{im}}$. Suppose that $w_{i1} < w$ for some i. For simplicity, assume that $w_{11} < w$, and w_{11} is a minimal element of $\{w_{j1} \mid w_{j1} < w\}$. Let us denote w_{11} by φ. Then for $i \geq 2$, $F_i|_{X_\varphi^v}$ is either 0, or is standard on X_φ^v. Hence restricting $(*)$ to X_φ^v, we obtain a nontrivial standard sum on X_φ^v being zero, which is not possible (by induction hypothesis). Hence we conclude that $w_{i1} = w$ for all i, $1 \leq i \leq m$. Canceling p_w, we obtain a linear relation among standard monomials on X_w^v of degree $m - 1$. Using induction on m, the required result follows. □

3.3. **Generation by Standard Monomials.**

Theorem 3.3.1. *Let $F = p_{w_1} \ldots p_{w_m}$ be any monomial in the Plücker coordinates of degree m. Then F is a linear combination of standard monomials of degree m.*

Proof. For $F = p_{w_1} \ldots p_{w_m}$, define

$$N_F = l(w_1)N^{m-1} + l(w_2)N^{m-2} + \ldots l(w_m),$$

where $N \gg 0$, say $N > d(n-d)$ $(= \dim G_{d,n})$ and $l(w) = \dim X_w$. If F is standard, there is nothing to prove. Let t be the first violation of standardness, i.e. $p_{w_1} \ldots p_{w_{t-1}}$

is standard, but $p_{w_1} \dots p_{w_t}$ is not. Hence $w_{t-1} \not\geq w_t$, and using the quadratic relations (cf. Theorem 1.0.1)

$$(*) \qquad p_{w_{t-1}} p_{w_t} = \sum_{\alpha,\beta} \pm p_\alpha p_\beta,$$

F can be expressed as $F = \sum F_i$, with $N_{F_i} > N_F$ (since $\alpha > w_{t-1}$ for all α on the right hand side of $(*)$). Now the required result is obtained by decreasing induction on N_F (the starting point of induction, i.e. the case when N_F is the largest, corresponds to standard monomial $F = p_\theta^m$, where $\theta = (n+1-d, n+2-d, \cdots, n)$, in which case F is clearly standard). □

Combining Theorems 3.2.1 and 3.3.1, we obtain

Theorem 3.3.2. *Standard monomials on X_w^v of degree m give a basis for R_w^v of degree m.*

As a consequence of Theorem 3.3.2 (or also Theorem 1.0.1), we have a qualitative description of a typical quadratic relation on a Richardson variety X_w^v as given by the following

Proposition 3.3.3. *Let $w, \tau, \varphi, v \in I_{d,n},\ w > \tau, \varphi$ and $\tau, \varphi > v$. Further let τ, φ be non-comparable (so that $p_\tau p_\varphi$ is a non-standard degree 2 monomial on X_w^v). Let*

$$(*) \qquad p_\tau p_\varphi = \sum_{\alpha,\beta} c_{\alpha,\beta} p_\alpha p_\beta,\ c_{\alpha,\beta} \in k^*$$

be the expression for $p_\tau p_\varphi$ as a sum of standard monomials on X_w^v. Then for every α, β on the right hand side we have, $\alpha >$ both τ and φ, and $\beta <$ both τ and φ.

Such a relation as in (*) is called a *straightening relation*.

3.4. **Equations Defining Richardson Varieties in the Grassmannian.** Let $w, v \in I_{d,n}$, with $w \geq v$. Let π_w^v be the map $R_0 \to R_w^v$ (the restriction map). Let $\ker \pi_w^v = J_w^v$. Let $Z_w^v = \{$all standard monomials $F \mid F$ contains some p_φ for some $w \not\geq \varphi$ or $\varphi \not\geq v\}$. We shall now give a set of generators for J_w^v in terms of Plücker coordinates.

Lemma 3.4.1. *Let $I_w^v = (p_\varphi, w \not\geq \varphi$ or $\varphi \not\geq v)$ (ideal in R_0). Then Z_w^v is a basis for I_w^v.*

Proof. Let $F \in I_w^v$. Then writing F as a linear combination of standard monomials

$$F = \sum a_i F_i + \sum b_j G_j,$$

where in the first sum each F_i contains some p_τ, with $w \not\geq \tau$ or $\tau \not\geq v$, and in the second sum each G_j contains only coordinates of the form p_τ, with $w \geq \tau \geq v$. This implies that $\sum a_iF_i \in I_w^v$, and hence we obtain

$$\sum b_jG_j \in I_w^v.$$

This now implies that considered as an element of R_w^v, $\sum b_jG_j$ is equal to 0 (note that $I_w^v \subset J_w^v$). Now the linear independence of standard monomials on X_w^v implies that $b_j = 0$ for all j. The required result now follows. □

Proposition 3.4.2. *Let $w, v \in I_{d,n}$ with $w \geq v$. Then $R_w^v = R_0/I_w^v$.*

Proof. We have, $R_w^v = R_0/J_w^v$ (where J_w^v is as above). We shall now show that the inclusion $I_w^v \subset J_w^v$ is in fact an equality. Let $F \in R_0$. Writing F as a linear combination of standard monomials

$$F = \sum a_iF_i + \sum b_jG_j,$$

where in the first sum each F_i contains some term p_τ, with $w \not\geq \tau$ or $\tau \not\geq v$, and in the second sum each G_j contains only coordinates p_τ, with $w \geq \tau \geq v$, we have, $\sum a_iF_i \in I_w^v$, and hence we obtain
$F \in J_w^v$
$\iff \sum b_jG_j \in J_w^v$ (since $\sum a_iF_i \in I_w^v$, and $I_w^v \subset J_w^v$)
$\iff \pi_w^v(F)$ $(= \sum b_jG_j)$ is zero
$\iff \sum b_jG_j$ (= a sum of standard monomials on X_w^v) is zero on X_w^v
$\iff b_j = 0$ for all j (in view of the linear independence of standard monomials on X_w^v)
$\iff F = \sum a_iF_i$
$\iff F \in I_w^v$.
Hence we obtain $J_w^v = I_w^v$. □

Equations defining Richardson varieties:

Let $w, v \in I_{d,n}$, with $w \geq v$. By Lemma 3.4.1 and Proposition 3.4.2, we have that the kernel of $(R_0)_1 \to (R_w^v)_1$ has a basis given by $\{p_\tau \mid w \not\geq \tau \text{ or } \tau \not\geq v\}$, and that the ideal J_w^v (= the kernel of the restriction map $R_0 \to R_w^v$) is generated by $\{p_\tau \mid w \not\geq \tau \text{ or } \tau \not\geq v\}$. Hence J_w^v is generated by the kernel of $(R_0)_1 \to (R_w^v)_1$. Thus we obtain that X_w^v is scheme-theoretically (even at the cone level) the intersection of $G_{d,n}$ with all hyperplanes in $\mathbb{P}(\wedge^d k^n)$ containing X_w^v. Further, as a closed subvariety of $G_{d,n}$, X_w^v is defined (scheme-theoretically) by the vanishing of $\{p_\tau \mid w \not\geq \tau \text{ or } \tau \not\geq v\}$.

4. Standard Monomial Theory for a Union of Richardson Varieties

In this section, we prove results similar to Theorems 3.2.1 and 3.3.2 for a union of Richardson varieties.

Let X_i be Richardson varieties in $G_{d,n}$. Let $X = \cup X_i$.

Definition 4.0.3. A monomial F in the Plücker coordinates is *standard* on the union $X = \cup X_i$ if it is standard on some X_i.

4.1. **Linear Independence of Standard Monomials on $X = \cup X_i$.**

Theorem 4.1.1. *Monomials standard on $X = \cup X_i$ are linearly independent.*

Proof. If possible, let

$$(*) \qquad 0 = \sum_{i=1}^{r} a_i F_i, \;\; a_i \in K^*$$

be a nontrivial relation among standard monomials on X. Suppose F_1 is standard on X_j. Then restricting $(*)$ to X_j, we obtain a nontrivial relation among standard monomials on X_j, which is a contradiction (note that for any i, $F_i|_{X_j}$ is either 0 or remains standard on X_j; further, $F_1|_{X_j}$ is non-zero). □

4.2. **Standard Monomial Basis.**

Theorem 4.2.1. *Let $X = \cup_{i=1}^{r} X_{w_i}^{v_i}$, and S the homogeneous coordinate ring of X. Then the standard monomials on X give a basis for S.*

Proof. For $w, v \in I_{d,n}$ with $w \geq v$, let I_w^v be as in Lemma 3.4.1. Let us denote $I_t = I_{w_t}^{v_t}, X_t = X_{w_t}^{v_t}$, $1 \leq t \leq r$. We have $R_{w_t}^{v_t} = R_0/I_t$ (cf. Proposition 3.4.2). Let $S = R_0/I$. Then $I = \cap I_t$ (note that being the intersection of radical ideals, I is also a radical ideal, and hence the set theoretic equality $X = \cup X_i$ is also scheme theoretic). A typical element in R_0/I may be written as $\pi(f)$, for some $f \in R_0$, where π is the canonical projection $R_0 \to R_0/I$. Let us write f as a sum of standard monomials

$$f = \sum a_j G_j + \sum b_l H_l,$$

where each G_j contains some p_{τ_j} such that $w_i \not\geq \tau_j$ or $\tau_j \not\geq v_i$, for $1 \leq i \leq r$; and for each H_l, there is some i_l, with $1 \leq i_l \leq r$, such that H_l is made up entirely of p_τ's with $w_{i_l} \geq \tau \geq v_{i_l}$. We have $\pi(f) = \sum b_l H_l$ (since $\sum a_j G_j \in I$). Thus we obtain that S (as a vector space) is generated by monomials standard on X. This together with the linear independence of standard monomials on X implies the required result. □

4.3. **Consequences.**

Theorem 4.3.1. *Let X_1, X_2 be two Richardson varieties in $G_{d,n}$. Then*
1. *$X_1 \cup X_2$ is reduced.*
2. *If $X_1 \cap X_2 \neq \emptyset$, then $X_1 \cap X_2$ is reduced.*

Proof. (1) Assertion is obvious.

(2) Let $X_1 = X_{w_1}^{v_1}$, $X_2 = X_{w_2}^{v_2}$, $I_1 = I_{w_1}^{v_1}$, and $I_2 = I_{w_2}^{v_2}$. Let A be the homogeneous coordinate ring of $X_1 \cap X_2$. Let $A = R_0/I$. Then $I = I_1 + I_2$. Let $F \in I$. Then by Lemma 3.4.1 and Proposition 3.4.2, in the expression for F as a linear combination of standard monomials

$$F = \sum a_j F_j,$$

each F_j contains some p_τ, where either $((w_1 \text{ or } w_2) \not\geq \tau)$ or $(\tau \not\geq (v_1 \text{ or } v_2))$. Let $X_1 \cap X_2 = X_\mu^\nu$ set theoretically, where $\mu = w_1 \wedge w_2$ and $\nu = v_1 \vee v_2$ (cf. Remark 2.2.3). If $B = R_0/\sqrt{I}$, then by Lemma 3.4.1 and Proposition 3.4.2, under $\pi : R_0 \to B$, $\ker \pi$ consists of all f such that $f = \sum c_k f_k$, f_k being standard monomials such that each f_k contains some p_φ, where $\mu \not\geq \varphi$ or $\varphi \not\geq \nu$. Hence either $((w_1 \text{ or } w_2) \not\geq \varphi)$ or $(\varphi \not\geq (v_1 \text{ or } v_2))$. Hence $\sqrt{I} = I$, and the required result follows from this. □

Definition 4.3.2. Let $w > v$. Define $\partial^+ X_w^v := \bigcup_{w > w' \geq v} X_{w'}^v$, and $\partial^- X_w^v := \bigcup_{w \geq v' > v} X_w^{v'}$.

Theorem 4.3.3. *(Pieri's formulas) Let $w > v$.*

(1) $X_w^v \cap \{p_w = 0\} = \partial^+ X_w^v$, *scheme theoretically.*
(2) $X_w^v \cap \{p_v = 0\} = \partial^- X_w^v$, *scheme theoretically.*

Proof. Let $X = \partial^+ X_w^v$, and let A be the homogeneous coordinate ring of X. Let $A = R_w^v/I$. Clearly, $(p_w) \subseteq I$, (p_w) being the principal ideal in R_w^v generated by p_w. Let $f \in I$. Writing f as

$$f = \sum b_i G_i + \sum c_j H_j,$$

where each G_i is a standard monomial in R_w^v starting with p_w and each H_j is a standard monomial in R_w^v starting with $p_{\theta_{j1}}$, where $\theta_{j1} < w$, we have, $\sum b_i G_i \in I$. This now implies $\sum c_j H_j$ is zero on $\partial^+ X_w^v$. But now $\sum c_j H_j$ being a sum of standard monomials on $\partial^+ X_w^v$, we have by Theorem 4.1.1, $c_j = 0$, for all j. Thus we obtain $f = \sum b_i G_i$, and hence $f \in (p_w)$. This implies $I = (p_w)$. Hence we obtain $A = R_w^v/(p_w)$, and (1) follows from this. The proof of (2) is similar. □

5. Vanishing Theorems

Let X be a union of Richardson varieties. Let $S(X, m)$ be the set of standard monomials on X of degree m, and $s(X, m)$ the cardinality of $S(X, m)$. If $X = X_w^v$ for some w, v, then $S(X, m)$ and $s(X, m)$ will also be denoted by just $S(w, v, m)$, respectively $s(w, v, m)$.

Lemma 5.0.4. (1) *Let $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are unions of Richardson varieties such that $Y_1 \cap Y_2 \neq \emptyset$. Then*

$$s(Y, m) = s(Y_1, m) + s(Y_2, m) - s(Y_1 \cap Y_2, m).$$

(2) *Let $w > v$. Then*

$$\begin{aligned} s(w, v, m) &= s(w, v, m-1) + s\left(\partial^+ X_w^v, m\right) \\ &= s(w, v, m-1) + s\left(\partial^- X_w^v, m\right) \end{aligned}$$

.

(1) and (2) are easy consequences of the results of the previous section.

Let X be a closed subvariety of $G_{d,n}$. Let $L = p^*(\mathcal{O}_{\mathbb{P}}(1))$, where $\mathbb{P} = \mathbb{P}(\wedge^d K^n)$, and $p : X \hookrightarrow \mathbb{P}$ is the Plücker embedding restricted to X.

Proposition 5.0.5. *Let r be an integer $\leq d(n-d)$. Suppose that all Richardson varieties X in $G_{d,n}$ of dimension at most r satisfy the following two conditions:*
(1) $H^i(X, L^m) = 0$, *for* $i \geq 1$, $m \geq 0$.
(2) *The set* $S(X, m)$ *is a basis for* $H^0(X, L^m)$, $m \geq 0$.
Then any union of Richardson varieties of dimension at most r which have nonempty intersection, and any nonempty intersection of Richardson varieties, satisfy (1) *and* (2).

Proof. The proof for intersections of Richardson varieties is clear, since any nonempty intersection of Richardson varieties is itself a Richardson variety (cf. Lemma 2.2.2 and Theorem 4.3.1).

We will prove the result for unions by induction on r. Let S_r denote the set of Richardson varieties X in $G_{d,n}$ of dimension at most r. Let $Y = \cup_{j=1}^{t} X_j,\ X_j \in S_r$. Let $Y_1 = \cup_{j=1}^{t-1} X_j$, and $Y_2 = X_t$. Consider the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0,$$

where $\mathcal{O}_Y \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2}$ is the map $f \mapsto (f|_{Y_1}, f|_{Y_2})$ and $\mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2}$ is the map $(f, g) \mapsto (f-g)|_{Y_1 \cap Y_2}$. Tensoring with L^m, we obtain the long exact sequence

$$\to H^{i-1}(Y_1 \cap Y_2, L^m) \to H^i(Y, L^m) \to H^i(Y_1, L^m) \oplus H^i(Y_2, L^m) \to H^i(Y_1 \cap Y_2, L^m) \to$$

Now $Y_1 \cap Y_2$ is reduced (cf. Theorem 4.3.1) and $Y_1 \cap Y_2 \in S_{r-1}$. Hence, by the induction hypothesis (1) and (2) hold for $Y_1 \cap Y_2$. In particular, if $m \geq 0$, then (2) implies that the map $H^0(Y_1, L^m) \oplus H^0(Y_2, L^m) \to H^0(Y_1 \cap Y_2, L^m)$ is surjective. Hence we obtain that the sequence

$$0 \to H^0(Y, L^m) \to H^0(Y_1, L^m) \oplus H^0(Y_2, L^m) \to H^0(Y_1 \cap Y_2, L^m) \to 0$$

is exact. This implies $H^0(Y_1 \cap Y_2, L^m) \to H^1(Y, L^m)$ is the zero map; we have, $H^1(Y, L^m) \to H^1(Y_1, L^m) \oplus H^1(Y_2, L^m)$ is also the zero map (since by induction $H^1(Y_1, L^m) = 0 = H^1(Y_2, L^m)$). Hence we obtain $H^1(Y, L^m) = 0,\ m \geq 0$, and for $i \geq 2$, the assertion that $H^i(Y, L^m) = 0,\ m \geq 0$ follows from the long exact cohomology sequence above (and induction hypothesis). This proves the assertion (1) for Y.

To prove assertion (2) for Y, we observe

$$\begin{aligned} h^0(Y, L^m) &= h^0(Y_1, L^m) + h^0(Y_2, L^m) - h^0(Y_1 \cap Y_2, L^m) \\ &= s(Y_1, m) + s(Y_2, m) - s(Y_1 \cap Y_2, m). \end{aligned}$$

Hence Lemma 5.0.4 implies that

$$h^0(Y, L^m) = s(Y, L^m).$$

This together with linear independence of standard monomials on Y proves assertion (2) for Y. □

Theorem 5.0.6. *Let X be a Richardson variety in $G_{d,n}$. Then*
(a) $H^i(X, L^m) = 0$ for $i \geq 1$, $m \geq 0$.
(b) $S(X, m)$ is a basis for $H^0(X, L^m)$, $m \geq 0$.

Proof. We prove the result by induction on m, and $\dim X$.

If $\dim X = 0$, X is just a point, and the result is obvious. Assume now that $\dim X \geq 1$. Let $X = X_w^v$, $w > v$. Let $Y = \partial^+ X_w^v$. Then by Pieri's formula (cf. §4.3.3), we have,

$$Y = X(\tau) \cap \{p_\tau = 0\} \quad \text{(scheme theoretically)}.$$

Hence the sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

is exact. Tensoring it with L^m, and writing the cohomology exact sequence, we obtain the long exact cohomology sequence

$$\cdots \to H^{i-1}(Y, L^m) \to H^i(X, L^{m-1}) \to H^i(X, L^m) \to H^i(Y, L^m) \to \cdots .$$

Let $m \geq 0$, $i \geq 2$. Then the induction hypothesis on $\dim X$ implies (in view of Proposition 5.0.5) that $H^i(Y, L^m) = 0$, $i \geq 1$. Hence we obtain that the sequence $0 \to H^i(X, L^{m-1}) \to H^i(X, L^m)$, $i \geq 2$, is exact. If $i = 1$, again the induction hypothesis implies the surjectivity of $H^0(X, L^m) \to H^0(Y, L^m)$. This in turn implies that the map $H^0(Y, L^m) \to H^1(X, L^{m-1})$ is the zero map, and hence we obtain that the sequence $0 \to H^1(X, L^{m-1}) \to H^1(X, L^m)$ is exact. Thus we obtain that $0 \to H^i(X, L^{m-1}) \to H^i(X, L^m)$, $m \geq 0$, $i \geq 1$ is exact. But $H^i(X, L^m) = 0$, $m \gg 0$, $i \geq 1$ (cf. [25]). Hence we obtain

$$H^i(X, L^m) = 0 \text{ for } i \geq 1,\ m \geq 0, \tag{1}$$

and

$$h^0(X, L^m) = h^0(X, L^{m-1}) + h^0(Y, L^m). \tag{2}$$

In particular, assertion (a) follows from (1). The induction hypothesis on m implies that $h^0(X, L^{m-1}) = s(X, m-1)$. On the other hand, the induction hypothesis on $\dim X$ implies (in view of Proposition 5.0.5) that $h^0(Y, L^m) = s(Y, m)$. Hence we obtain

$$h^0(X, L^m) = s(X, m-1) + s(Y, m). \tag{3}$$

Now (3) together with Lemma 5.0.4,(2) implies $h^0(X, L^m) = s(X, m)$. Hence (b) follows in view of the linear independence of standard monomials on X_w^v (cf. Theorem 3.2.1). □

Corollary 5.0.7. *We have*
1. $R_w^v = \bigoplus_{m \in \mathbb{Z}^+} H^0(X_w^v, L^m),\ w \geq v.$
2. $\dim H^0(\partial^+ X_w^v, L^m) = \dim H^0(\partial^- X_w^v, L^m)$, $w > v$, $m \geq 0$.

Proof. Assertion 1 follows immediately from Theorems 3.3.2 and 5.0.6(b). Assertion 2 follows from Lemma 5.0.4, Theorem 5.0.5(2), and Theorem 5.0.6(b). □

6. Tangent Space and Smoothness

6.1. **The Zariski Tangent Space.** Let x be a point on a variety X. Let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathcal{O}_{X,\,x}$ with residue field $K(x)(=\mathcal{O}_{X,\,x}/\mathfrak{m}_x)$. Note that $K(x)=K$ (since K is algebraically closed). Recall that the Zariski tangent space to X at x is defined as

$$\begin{aligned} T_x(X) &= \mathrm{Der}_K(\mathcal{O}_{X,\,x}, K(x)) \\ &= \{D : \mathcal{O}_{X,\,x} \to K(x),\ K\text{-linear such that } D(ab) = D(a)b + aD(b)\} \end{aligned}$$

(here $K(x)$ is regarded as an $\mathcal{O}_{X,\,x}$-module). It can be seen easily that $T_x(X)$ is canonically isomorphic to $\mathrm{Hom}_{K\text{-mod}}(\mathfrak{m}_x/\mathfrak{m}_x^2, K)$.

6.2. **Smooth and Non-smooth Points.** A point x on a variety X is said to be a *simple* or *smooth* or *nonsingular point of* X if $\mathcal{O}_{X,\,x}$ is a regular local ring. A point x which is not simple is called a *multiple* or *non-smooth* or *singular point* of X. The set $\mathrm{Sing}\,X = \{x \in X \mid x \text{ is a singular point}\}$ is called the *singular locus of* X. A variety X is said to be *smooth* if $\mathrm{Sing}X = \emptyset$. We recall the well known

Theorem 6.2.1. *Let $x \in X$. Then $\dim_K T_x(X) \geq \dim \mathcal{O}_{X,\,x}$ with equality if and only if x is a simple point of X.*

6.3. **The Space $T^v_{w,\tau}$.** Let $G, T, B, P_d, W, R, W_{P_d}, R_{P_d}$ etc., be as in §1.2. We shall henceforth denote P_d by just P. For $\alpha \in R$, let X_α be the element of the Chevalley basis for $\mathfrak{g}$ $(=\mathrm{Lie}G)$, corresponding to α. We follow [2] for denoting elements of R, R^+ etc.

For $w \geq \tau \geq v$, let $T^v_{w,\tau}$ be the Zariski tangent space to X^v_w at e_τ. Let w_0 be the element of largest length in W. Now the tangent space to G at e_{id} is $\mathfrak{g}$, and hence the tangent space to G/P at e_{id} is $\bigoplus_{\beta \in R^+ \setminus R^+_P} \mathfrak{g}_{-\beta}$. For $\tau \in W$, identifying G/P with $G/{}^\tau P$ (where ${}^\tau P = \tau P \tau^{-1}$) via the map $gP \mapsto (n_\tau g n_\tau^{-1}) {}^\tau P$, n_τ being a fixed lift of τ in $N_G(T)$, we have, the tangent space to G/P at e_τ is $\bigoplus_{\beta \in \tau(R^+) \setminus \tau(R^+_P)} \mathfrak{g}_{-\beta}$, i.e.,

$$T^{\mathrm{id}}_{w_0,\tau} = \bigoplus_{\beta \in \tau(R^+) \setminus \tau(R^+_P)} \mathfrak{g}_{-\beta}.$$

Set

$$N^v_{w,\tau} = \{\beta \in \tau(R^+) \setminus \tau(R^+_P) \mid X_{-\beta} \in T^v_{w,\tau}\}.$$

Since $T^v_{w,\tau}$ is a T-stable subspace of $T^v_{w_0,\tau}$, we have

$$T^v_{w,\tau} = \text{ the span of } \{X_{-\beta},\ \beta \in N^v_{w,\tau}\}.$$

6.4. **Certain Canonical Vectors in $T^v_{w,\tau}$.** For a root $\alpha \in R^+ \setminus R^+_P$, let Z_α denote the $SL(2)$-copy in G corresponding to α; note that Z_α is simply the subgroup of G generated by U_α and $U_{-\alpha}$. Given $x \in W^P$, precisely one of $\{U_\alpha, U_{-\alpha}\}$ fixes the point e_x. Thus $Z_\alpha \cdot e_x$ is a T-stable curve in G/P (note that $Z_\alpha \cdot e_x \cong \mathbb{P}^1$), and conversely any T-stable curve in G/P is of this form (cf. [5]). Now a T-stable curve $Z_\alpha \cdot e_x$ is contained in a Richardson variety X^v_w if and only if $e_x, e_{s_\alpha x}$ are both in X^v_w.

Lemma 6.4.1. *Let $w, \tau, v \in W^P$, $w \geq \tau \geq v$. Let $\beta \in \tau(R^+ \setminus R^+_P)$. If $w \geq s_\beta \tau \geq v \,(\mathrm{mod}\, W_P)$, then $X_{-\beta} \in T^v_{w,\tau}$.*

(Note that $s_\beta\tau$ need not be in W^P.)

Proof. The hypothesis that $w \geq s_\beta\tau \geq v(\mathrm{mod}\, W_P)$ implies that the curve $Z_\beta \cdot e_\tau$ is contained in X^v_w. Now the tangent space to $Z_\beta \cdot e_\tau$ at e_τ is the one-dimensional span of $X_{-\beta}$. The required result now follows. □

We shall show in Theorem 6.7.2 that w, τ, v being as above, $T^v_{w,\tau}$ is precisely the span of $\{X_{-\beta},\ \beta \in \tau(R^+ \setminus R^+_P) \mid w \geq s_\beta\tau \geq v \,(\mathrm{mod}\, W_P)\}$.

6.5. **A Canonical Affine Neighborhood of a T-fixed Point.** Let $\tau \in W$. Let U^-_τ be the unipotent subgroup of G generated by the root subgroups $U_{-\beta},\ \beta \in \tau(R^+)$ (note that U^-_τ is the unipotent part of the Borel sub group ${}^\tau B^-$, opposite to ${}^\tau B$ $(= \tau B \tau^{-1})$). We have

$$U_{-\beta} \cong \mathbb{G}_a,\ U^-_\tau \cong \prod_{\beta \in \tau(R^+)} U_{-\beta}.$$

Now, U^-_τ acts on G/P by left multiplication. The isotropy subgroup in U^-_τ at e_τ is $\Pi_{\beta \in \tau(R^+_P)} U_{-\beta}$. Thus $U^-_\tau e_\tau \cong \Pi_{\beta \in \tau(R^+ \setminus R^+_P)} U_{-\beta}$. In this way, $U^-_\tau e_\tau$ gets identified with $\mathbb{A}^N$, where $N = \#(R^+ \setminus R^+_P)$. We shall denote the induced coordinate system on $U^-_\tau e_\tau$ by $\{x_{-\beta},\ \beta \in \tau(R^+ \setminus R^+_P)\}$. In the sequel, we shall denote $U^-_\tau e_\tau$ by $\mathcal{O}^-_\tau$ also. Thus we obtain that $\mathcal{O}^-_\tau$ is an affine neighborhood of e_τ in G/P.

6.6. **The Affine Variety $Y^v_{w,\tau}$.** For $w, \tau, v \in W$, $w \geq \tau \geq v$, let us denote $Y^v_{w,\tau} := \mathcal{O}^-_\tau \cap X^v_{w,\tau}$. It is a nonempty affine open subvariety of X^v_w, and a closed subvariety of the affine space $\mathcal{O}^-_\tau$.

Note that L, the ample generator of $\mathrm{Pic}(G/P)$, is the line bundle corresponding to the Plücker embedding, and $H^0(G/P, L) = (\wedge^d K^n)^*$, which has a basis given by the Plücker coordinates $\{p_\theta,\ \theta \in I_{d,n}\}$. Note also that the affine ring $\mathcal{O}^-_\tau$ may be identified as the homogeneous localization $(R_0)_{(p_\tau)}$, R_0 being as in §3.1. We shall denote p_θ/p_τ by $f_{\theta,\tau}$. Let $I^v_{w,\tau}$ be the ideal defining $Y^v_{w,\tau}$ as a closed subvariety of $\mathcal{O}^-_\tau$. Then $I^v_{w,\tau}$ is generated by $\{f_{\theta,\tau} \mid w \not\geq \theta \text{ or } \theta \not\leq v\}$.

6.7. **Basis for Tangent Space & Criterion for Smoothness of X_w^v at e_τ.** Let Y be an affine variety in $\mathbb{A}^n$, and let $I(Y)$ be the ideal defining Y in $\mathbb{A}^n$. Let $I(Y)$ be generated by $\{f_1, f_2, \ldots, f_r\}$. Let J be the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$. We have (cf. Theorem 6.2.1) the dimension of the tangent space to Y at a point P is greater than or equal to the dimension of Y, with equality if and only if P is a smooth point; equivalently, rank $J_P \leq \mathrm{codim}_{\mathbb{A}^n} Y$ with equality if and only if P is a smooth point of Y (here J_P denotes J evaluated at P).

Let $w, \tau, v \in W,\ w \geq \tau \geq v$. The problem of determining whether or not e_τ is a smooth point of X_w^v is equivalent to determining whether or not e_τ is a smooth point of $Y_{w,\tau}^v$ (since $Y_{w,\tau}^v$ is an open neighborhood of e_τ in X_w^v). In view of Jacobian criterion, the problem is reduced to computing $(\partial f_{\theta,\tau}/\partial x_{-\beta})_{e_\tau}$, $w \not\geq \theta$ or $\theta \not\geq v$ (the Jacobian matrix evaluated at e_τ). To carry out this computation, we first observe the following:

Let V be the G-module $H^0(G/P, L)$ $(= (\wedge^d K^n)^*)$. Now V is also a $\mathfrak{g}$-module. Given X in $\mathfrak{g}$, we identify X with the corresponding right invariant vector field D_X on G. Thus we have $D_X p_\theta = X p_\theta$, and we note that

$$(\partial f_{\theta,\tau}/\partial x_{-\beta})(e_\tau) = X_{-\beta}\, p_\theta(e_\tau),\ \beta \in \tau(R^+ \setminus R_P^+),$$

where the left hand side denotes the partial derivative evaluated at e_τ.

We make the folowing three observations:

(1) For $\theta, \mu \in W^P$, $p_\theta(e_\mu) \neq 0 \iff \theta = \mu$, where, recall that for $\theta = (i_1 \cdots i_d) \in W^P$, e_θ denotes the vector $e_{i_1} \wedge \cdots \wedge e_{i_d}$ in $\wedge^d K^n$, and p_θ denotes the Plücker coordinate associated to θ.

(2) Let X_α be the element of the Chevalley basis of $\mathfrak{g}$, corresponding to $\alpha \in R$. If $X_\alpha p_\mu \neq 0$, $\mu \in W^P$, then $X_\alpha p_\mu = \pm p_{s_\alpha \mu}$, where s_α is the reflection corresponding to the root α.

(3) For $\alpha \neq \beta$, if $X_\alpha p_\mu,\ X_\beta p_\mu$ are non-zero, then $X_\alpha p_\mu \neq X_\beta p_\mu$.

The first remark is obvious, since $\{p_\theta \mid \theta \in W^P\}$ is the basis of $(\wedge^P K^n)^*$ $(= H^0(G/P, L))$, dual to the basis $\{e_\varphi,\ \varphi \in I_{d,n}\}$ of $\wedge^d K^n$. The second remark is a consequence of SL_2 theory, using the following facts:

(a) $|\langle \chi, \alpha^* \rangle| = \left| \frac{2(\chi,\alpha)}{(\alpha,\alpha)} \right| = 0$ or 1, χ being the weight of p_μ.

(b) p_μ is the lowest weight vector for the Borel subgroup ${}^\mu B = \mu B \mu^{-1}$.

The third remark follows from weight considerations (note that if $X_\alpha p_\mu \neq 0$, then $X_\alpha p_\mu$ is a weight vector (for the T-action) of weight $\chi + \alpha$, χ being the weight of p_μ).

Theorem 6.7.1. *Let $w, \tau, v \in W^P,\ w \geq \tau \geq v$. Then*

$$\dim T_{w,\tau}^v = \#\{\gamma \in \tau(R^+ \setminus R_P^+) \mid w \geq s_\gamma \tau \geq v \,(\mathrm{mod}\, W_P)\}.$$

Proof. By Lemma 3.4.1 and Proposition 3.4.2, we have, $I_{w,\tau}^v$ is generated by $\{f_{\theta,\tau} \mid w \not\geq \theta$ or $\theta \not\geq v\}$. Denoting the affine coordinates on $\mathcal{O}_\tau^-$ by $x_{-\beta}$, $\beta \in \tau(R^+ \setminus R_P^+)$, we have the evaluations of $\frac{\partial f_{\theta,\tau}}{\partial x_\beta}$ and $X_\beta p_\theta$ at e_τ coincide. Let J_w^v denote the Jacobian matrix of $Y^v_{w,\tau}$ (considered as a subvariety of the affine space $\mathcal{O}^-_\tau$). We shall index the rows of J^v_w by $\{f_{\theta,\tau} \mid w \not\geq \theta \text{ or } \theta \not\geq v\}$ and the columns by $x_{-\beta},\ \beta \in \tau(R^+ \setminus R^+_P)$. Let $J^v_w(\tau)$ denote J^v_w evaluated at e_τ. Now in view of (1) & (2) above, the $(f_{\theta,\tau}, x_{-\beta})$-th entry in $J^v_w(\tau)$ is non-zero if and only if $X_\beta p_\theta = \pm p_\tau$. Hence in view of (3) above, we obtain that in each row of $J^v_w(\tau)$, there is at most one non-zero entry. Hence $\mathrm{rank} J^v_w(\tau)$ = the number of non-zero columns of $J^v_w(\tau)$. Now, $X_\beta p_\theta = \pm p_\tau$ if and only if $\theta \equiv s_\beta\tau \,(\mathrm{mod}\, W_P)$. Thus the column of $J^v_w(\tau)$ indexed by $x_{-\beta}$ is non-zero if and only if $w \not\geq s_\beta\tau \,(\mathrm{mod}\, W_P)$ or $s_\beta\tau \not\geq v \,(\mathrm{mod}\, W_P)$. Hence $\mathrm{rank} J^v_w(\tau) = \#\{\gamma \in \tau(R^+ \setminus R^+_P) \mid w \not\geq s_\gamma\tau \,(\mathrm{mod}\, W_P) \text{ or } s_\gamma\tau \not\geq v \,(\mathrm{mod}\, W_P)\}$ and thus we obtain

$$\dim T^v_{w,\tau} = \#\{\gamma \in \tau(R^+ \setminus R^+_P) \mid w \geq s_\gamma\tau \geq v \,(\mathrm{mod}\, W_P)\}.$$

□

Theorem 6.7.2. *Let w, τ, v be as in Theorem 6.7.1. Then $\{X_{-\beta},\ \beta \in \tau(R^+ \setminus R^+_P) \mid w \geq s_\beta\tau \geq v \,(\mathrm{mod}\, W_P)\}$ is a basis for $T^v_{w,\tau}$.*

Proof. Let $\beta \in \tau(R^+ \setminus R^+_P)$ be such that $w \geq s_\beta\tau \geq v \,(\mathrm{mod}\, W_P)$. We have (by Lemma 6.4.1), $X_{-\beta} \in T^v_{w,\tau}$. On the other hand, by Theorem 6.7.1,
$\dim T^v_{w,\tau} = \#\{\beta \in \tau(R^+ \setminus R^+_P) \mid w \geq s_\beta\tau \geq v \,(\mathrm{mod}\, W_P)\}$. The result follows from this. □

Corollary 6.7.3. *X^v_w is smooth at e_τ if and only if $l(w) - l(v) = \#\{\alpha \in R^+ \setminus R^+_P \mid w \geq \tau s_\alpha \geq v \,(\mathrm{mod}\, W_P)\}$.*

Proof. We have, X^v_w is smooth at e_τ if and only if $\dim T^v_{w,\tau} = \dim X^v_w$, and the result follows in view of Corollary 2.1.2 and Theorem 6.7.1 (note that if $\beta = \tau(\alpha)$, then $s_\beta\tau = \tau s_\alpha \,(\mathrm{mod}\, W_P)$). □

7. Multiplicity at a Singular Point

7.1. **Multiplicity of an Algebraic Variety at a Point.** Let B be a graded, affine K-algebra such that B_1 generates B (as a K-algebra). Let $X = \mathrm{Proj}(B)$. The function $h_B(m)$ (or $h_X(m)$) $= \dim_K B_m,\ m \in \mathbb{Z}$ is called the *Hilbert function* of B (or X). There exists a polynomial $P_B(x)$ (or $P_X(x)$) $\in \mathbb{Q}[x]$, called the *Hilbert polynomial* of B (or X), such that $f_B(m) = P_B(m)$ for $m \gg 0$. Let r denote the degree of $P_B(x)$. Then $r = \dim(X)$, and the leading coefficient of $P_B(x)$ is of the form $c_B/r!$, where $c_B \in \mathbb{N}$. The integer c_B is called the *degree of* X, and denoted $\deg(X)$ (see [7] for details). In the sequel we shall also denote $\deg(X)$ by $\deg(B)$.

Let X be an algebraic variety, and let $P \in X$. Let $A = \mathcal{O}_{X,P}$ be the stalk at P and $\mathfrak{m}$ the unique maximal ideal of the local ring A. Then the *tangent cone* to X at P, denoted $\mathrm{TC}_P(X)$, is $\mathrm{Spec}(\mathrm{gr}(A, \mathfrak{m}))$, where $\mathrm{gr}(A, \mathfrak{m}) = \oplus_{j=0}^\infty \mathfrak{m}^j/\mathfrak{m}^{j+1}$. The *multiplicity* of X at P, denoted $\mathrm{mult}_P(X)$, is $\deg(\mathrm{Proj}(\mathrm{gr}(A, \mathfrak{m})))$. (If $X \subset K^n$ is an affine closed subvariety, and $m_P \subset K[X]$ is the maximal ideal corresponding to $P \in X$, then $\mathrm{gr}(K[X], m_P) = \mathrm{gr}(A, \mathfrak{m})$.)

7.2. **Evaluation of Plücker Coordinates on $U_\tau^- e_\tau$.** Let $X = X_w^v$. Consider a $\tau \in W^P$ such that $w \geq \tau \geq v$.
I. Let us first consider the case $\tau = \text{id}$. We identify $U^- e_{\text{id}}$ with

$$\left\{ \begin{pmatrix} & \text{Id}_{d\times d} & \\ x_{d+1\,1} & \cdots & x_{d+1\,d} \\ \vdots & & \vdots \\ x_{n\,1} & \cdots & x_{n\,d} \end{pmatrix}, \quad x_{ij} \in k, \quad d+1 \leq i \leq n, 1 \leq j \leq d \right\}.$$

Let A be the affine algebra of $U^- e_{\text{id}}$. Let us identify A with the polynomial algebra $k[x_{-\beta}, \beta \in R^+ \setminus R_P^+]$. To be very precise, we have $R^+ \setminus R_P^+ = \{\epsilon_j - \epsilon_i,\ 1 \leq j \leq d,\ d+1 \leq i \leq n\}$; given $\beta \in R^+ \setminus R_P^+$, say $\beta = \epsilon_j - \epsilon_i$, we identify $x_{-\beta}$ with x_{ij}. Hence we obtain that the expression for $f_{\theta,\text{id}}$ in the local coordinates $x_{-\beta}$'s is homogeneous.

Example 7.2.1. Consider $G_{2,4}$. Then

$$U^- e_{\text{id}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix}, \quad x_{ij} \in k \right\}.$$

On $U^- e_{\text{id}}$, we have $p_{12} = 1$, $p_{13} = x_{32}$, $p_{14} = x_{42}$, $p_{23} = x_{31}$, $p_{24} = x_{41}$, $p_{34} = x_{31}x_{42} - x_{41}x_{32}$.

Thus a Plücker coordinate is homogeneous in the local coordinates $x_{ij},\ d+1 \leq i \leq n,\ 1 \leq j \leq d$.
II. Let now τ be any other element in W^P, say $\tau = (a_1, \ldots, a_n)$. Then $U_\tau^- e_\tau$ consists of $\{N_{d,n}\}$, where $N_{d,n}$ is obtained from $\begin{pmatrix} \text{Id} \\ X \end{pmatrix}_{n\times d}$ (with notations as above) by permuting the rows by τ^{-1}. (Note that $U_\tau^- e_\tau = \tau U^- e_{\text{id}}$.)

Example 7.2.2. Consider $G_{2,4}$, and let $\tau = (2314)$. Then $\tau^{-1} = (3124)$, and

$$U_\tau^- e_\tau = \left\{ \begin{pmatrix} x_{31} & x_{32} \\ 1 & 0 \\ 0 & 1 \\ x_{41} & x_{42} \end{pmatrix}, \quad x_{ij} \in k \right\}.$$

We have on $U_\tau^- e_\tau$, $p_{12} = -x_{32}$, $p_{13} = x_{31}$, $p_{14} = x_{31}x_{42} - x_{41}x_{32}$, $p_{23} = 1$, $p_{24} = x_{42}$, $p_{34} = -x_{41}$.

As in the case $\tau = \text{id}$, we find that for $\theta \in W^P$, $f_{\theta,\tau} := p_\theta|_{U_\tau^- e_\tau}$ is homogeneous in local coordinates. In fact we have

Proposition 7.2.3. *Let $\theta \in W^P$. We have a natural isomorphism*

$$k[x_{-\beta}, \beta \in R^+ \setminus R_P^+] \cong k[x_{-\tau(\beta)}, \beta \in R^+ \setminus R_P^+],$$

given by

$$f_{\theta,id} \mapsto f_{\tau\theta,\tau}.$$

The proof is immediate from the above identifications of $U^- e_{\rm id}$ and $U_\tau^- e_\tau$. As a consequence, we have

Corollary 7.2.4. *Let $\theta \in W^P$. Then the polynomial expression for $f_{\theta,\tau}$ in the local coordinates $\{x_{-\tau(\beta)}, \beta \in R^+ \setminus R_P^+\}$ is homogeneous.*

7.3. **The algebra $A^v_{w,\tau}$.** As above, we identify A_τ, the affine algebra of $U_\tau^- e_\tau$ with the polynomial algebra $K[x_{-\beta}, \beta \in \tau(R^+ \setminus R_P^+)]$. Let $A^v_{w,\tau} = A_\tau / I^v_{w,\tau}$, where $I^v_{w\tau}$ is the ideal of elements of A_τ that vanish on $X^v_w \cap U^-_\tau e_\tau$.

Now $I(X^v_w)$, the ideal of X^v_w in G/P, is generated by $\{p_\theta, \theta \in W^P \mid w \not\geq \theta \text{ or } \theta \not\geq v\}$. Hence we obtain (cf. Corollary 7.2.4) that $I^v_{w,\tau}$ is homogeneous. Hence we get

$$(*) \qquad \mathrm{gr}\,(A^v_{w,\tau}, M^v_{w,\tau}) = A^v_{w,\tau},$$

where $M^v_{w,\tau}$ is the maximal ideal of $A^v_{w,\tau}$ corresponding to e_τ. In particular, denoting the image of $x_{-\beta}$ under the canonical map $A_\tau \to A^v_{w,\tau}$ by just $x_{-\beta}$, the set $\{x_{-\beta} \mid \beta \in \tau(R^+ \setminus R_P^+)\}$ generates $A^v_{w,\tau}$. Let R^v_w be the homogeneous coordinate ring of X^v_w (for the Plücker embedding), $Y^v_{w,\tau} = X^v_w \cap U^-_\tau e_\tau$. Then $K[Y^v_{w,\tau}] = A^v_{w,\tau}$ gets identified with the homogeneous localization $(R^v_w)_{(p_\tau)}$, i.e. the subring of $(R^v_w)_{p_\tau}$ (the localization of R^v_w with respect to p_τ) generated by the elements

$$\{\frac{p_\theta}{p_\tau}, \theta \in W^P, w \geq \theta \geq v\}.$$

7.4. **The Integer $\deg_\tau(\theta)$.** Let $\theta \in W^P$. We define $\deg_\tau(\theta)$ by

$$\deg_\tau(\theta) := \deg f_{\theta,\tau}$$

(note that $f_{\theta,\tau}$ is homogeneous, cf. Corollary 7.2.4). In fact, we have an explicit expression for $\deg_\tau(\theta)$, as follows (cf. [13]):

Proposition 7.4.1. *Let $\theta \in W^P$. Let $\tau = (a_1, \dots, a_n)$, $\theta = (b_1, \dots, b_n)$. Let $r = \#\{a_1, \dots, a_d\} \cap \{b_1, \dots, b_d\}$. Then $\deg_\tau(\theta) = d - r$.*

7.5. **A Basis for the Tangent Cone.** Let $Z_\tau = \{\theta \in W^P \,|\, \text{either } \theta \geq \tau \text{ or } \tau \geq \theta\}$.

Theorem 7.5.1. *With notations as above, given $r \in \mathbb{Z}^+$,*

$$\{f_{\theta_1,\tau} \dots f_{\theta_m,\tau} \mid w \geq \theta_1 \geq ... \geq \theta_m \geq v,\, \theta_i \in Z_\tau,\, \sum_{i=1}^m \deg_\tau(\theta_i) = r\}$$

is a basis for $(M^v_{w,\tau})^r/(M^v_{w,\tau})^{r+1}$.

Proof. For $F = p_{\theta_1}\cdots p_{\theta_m}$, let deg$F$ denote the degree of $f_{\theta_1,\tau}\cdots f_{\theta_m,\tau}$. Let

$$A_r = \{F = p_{\theta_1}\cdots p_{\theta_m},\ w \geq \theta_i \geq v \,|\, \deg F = r\}.$$

Then in view of the relation (*) in §7.3, we have, A_r generates $(M^v_{w,\tau})^r/(M^v_{w,\tau})^{r+1}$. Let $F \in A_r$, say $F = p_{\tau_1}\cdots p_{\tau_m}$. From the results in §3, we know that $p_{\tau_1}\cdots p_{\tau_m}$ is a linear combination of standard monomials $p_{\theta_1}\cdots p_{\theta_m},\ w \geq \theta_i \geq v$. We *claim* that in each $p_{\theta_1}\cdots p_{\theta_m},\ \theta_i \in Z_\tau$, for all i. Suppose that for some $i, \theta_i \notin Z_\tau$. This means θ_i and τ are not comparable. Then using the fact that $f_{\tau,\tau} = 1$, on $Y^v_{w,\tau}$, we replace p_{θ_i} by $p_{\theta_i}p_\tau$ in $p_{\theta_1}\cdots p_{\theta_m}$. We now use the straightening relation (cf. Proposition 3.3.3) $p_{\theta_i}p_\tau = \sum c_{\alpha,\beta}p_\alpha p_\beta$ on X^v_w, where in each term $p_\alpha p_\beta$ on the right hand side, we have $\alpha < w$, and $\alpha >$ both θ_i and τ, and $\beta <$ both θ_i and τ; in particular, we have, in each term $p_\alpha p_\beta$ on the right hand side, α, β belong to Z_τ. We now proceed as in the proof of Theorem 3.3.1 to conclude that on $Y^v_{w,\tau}$, F is a linear combination of standard monomials in p_θ's, $\theta \in Z_\tau$ which proves the Claim.

Clearly, $\{f_{\theta_1,\tau}\dots f_{\theta_m,\tau} \mid w \geq \theta_1 \geq \dots \geq \theta_m \geq v,\ \theta_i \in Z_\tau\}$ is linearly independent in view of Theorem 3.2.1 (Since $p^l_\tau f_{\theta_1,\tau}\cdots f_{\theta_m,\tau} = p^{l-m}_\tau p_{\theta_1}\cdots p_{\theta_m}$ for $l \geq m$, and the monomial on the right hand side is standard since $\theta_i \in Z_\tau$). □

7.6. Recursive Formulas for $\mathrm{mult}_\tau X^v_w$.

Definition 7.6.1. If $w > \tau \geq v$, define $\partial^{v,+}_{w,\tau} := \{w' \in W^P \mid w > w' \geq \tau \geq v,\ l(w') = l(w) - 1\}$. If $w \geq \tau > v$, define $\partial^{v,-}_{w,\tau} := \{v' \in W^P \mid w \geq \tau \geq v' > v, l(v') = l(v) + 1\}$.

Theorem 7.6.2. (1) *Suppose* $w > \tau \geq v$. *Then*

$$(\mathrm{mult}_\tau X^v_w)\deg_\tau w = \sum_{w' \in \partial^{v,+}_{w,\tau}} \mathrm{mult}_\tau X^v_{w'}.$$

(2) *Suppose* $w \geq \tau > v$. *Then*

$$(\mathrm{mult}_\tau X^v_w)\deg_\tau v = \sum_{v' \in \partial^{v,-}_{w,\tau}} \mathrm{mult}_\tau X^{v'}_w.$$

(3) $\mathrm{mult}_\tau X^\tau_\tau = 1$.

Proof. Since X^τ_τ is a single point, (3) is trivial. We will prove (1); the proof of (2) is similar.

Let $H_\tau = \bigcup_{w' \in \partial^{v,+}_{w,\tau}} X^v_{w'}$. Let $\varphi^v_w(r)$ (resp. $\varphi_{H_\tau}(r)$) be the Hilbert function for the tangent cone of X^v_w (resp. H_τ) at $e(\tau)$, i.e.

$$\varphi^v_w(r) = \dim((M^v_{w,\tau})^r/(M^v_{w,\tau})^{r+1}).$$

Let

$$\mathcal{B}^v_{w,\tau}(r) = \left\{p_{\tau_1}\dots p_{\tau_m},\ \tau_i \in Z_\tau \mid (1)\, w \geq \tau_1 \geq \cdots \geq \tau_m \geq v, (2) \sum \deg_\tau(\tau_i) = r\right\}.$$

Let

$$\mathcal{B}_1 = \{p_{\tau_1} \dots p_{\tau_m} \in \mathcal{B}^v_{w,\tau}(r) \mid \tau_1 = w\},$$
$$\mathcal{B}_2 = \{p_{\tau_1} \dots p_{\tau_m} \in \mathcal{B}^v_{w,\tau}(r) \mid \tau_1 < w\}.$$

We have $\mathcal{B}^v_{w,\tau}(r) = \mathcal{B}_1 \dot\cup \mathcal{B}_2$. Hence denoting $\deg_\tau(w)$ by d, we obtain

$$\varphi^v_w(r+d) = \varphi^v_w(r) + \varphi_{H_\tau}(r+d).$$

Taking $r \gg 0$ and comparing the coefficients of r^{u-1}, where $u = \dim X^v_w$, we obtain the result. □

Corollary 7.6.3. *Let $w > \tau > v$. Then*

$$(\operatorname{mult}_\tau X^v_w)(\deg_\tau w + \deg_\tau v) = \sum_{w' \in \partial^{v,+}_{w,\tau}} \operatorname{mult}_\tau X^v_{w'} + \sum_{v' \in \partial^{v,-}_{w,\tau}} \operatorname{mult}_\tau X^{v'}_w.$$

Theorem 7.6.4. *Let $w \geq \tau \geq v$. Then* $\operatorname{mult}_\tau X^v_w = (\operatorname{mult}_\tau X_w) \cdot (\operatorname{mult}_\tau X^v)$

Proof. We proceed by induction on $\dim X^v_w$.

If $\dim X^v_w = 0$, then $w = \tau = v$. In this case, by Theorem 7.6.2 (3), we have that $\operatorname{mult}_\tau X^\tau_\tau = 1$. Since $e_\tau \in Be_\tau \subseteq X_w$, and Be_τ is an affine space open in X_w, e_τ is a smooth point of X_w, i.e. $\operatorname{mult}_\tau X_w = 1$. Similarly, $\operatorname{mult}_\tau X^v = 1$.

Next suppose that $\dim X^v_w > 0$, and $w > \tau \geq v$. By Theorem 7.6.2 (1),

$$\begin{aligned}
\operatorname{mult}_\tau X^v_w &= \frac{1}{\deg_\tau w} \sum_{w' \in \partial^{v,+}_{w,\tau}} \operatorname{mult}_\tau X^v_{w'} \\
&= \frac{1}{\deg_\tau w} \sum_{w' \in \partial^{v,+}_{w,\tau}} \operatorname{mult}_\tau X_{w'} \cdot \operatorname{mult}_\tau X^v \\
&= \left(\frac{1}{\deg_\tau w} \sum_{w' \in \partial^{v,+}_{w,\tau}} \operatorname{mult}_\tau X_{w'} \right) \cdot \operatorname{mult}_\tau X^v \\
&= \left(\frac{1}{\deg_\tau w} \sum_{w' \in \partial^{v,+}_{w,\tau}} \operatorname{mult}_\tau X^{\mathrm{id}}_{w'} \right) \cdot \operatorname{mult}_\tau X^v \\
&= \left(\operatorname{mult}_\tau X^{\mathrm{id}}_w \right) \cdot \operatorname{mult}_\tau X^v = \operatorname{mult}_\tau X_w \cdot \operatorname{mult}_\tau X^v.
\end{aligned}$$

The case of $\dim X^v_w > 0$ and $w = \tau > v$ is proven similarly. □

Corollary 7.6.5. *Let $w \geq \tau \geq v$. Then X^v_w is smooth at e_τ if and only if both X_w and X^v are smooth at e_τ.*

Remark 7.6.6. The following alternate proof of Theorem 7.6.4 is due to the referee, and we thank the referee for the same.

Identify $\mathcal{O}_\tau^-$ with the affine space $\mathbb{A}^N$ where $N = d(n-d)$, by the coordinate functions defined in §6.5. Then $X_w \cap \mathcal{O}_\tau^-$ and $X^v \cap \mathcal{O}_\tau^-$ are closed subvarieties of that affine space, both invariant under scalar multiplication (e.g. by Corollary 7.2.4). Moreover, X_w and X^v intersect properly along the irreducible subvariety X_w^v; in addition, the Schubert cells C_w and C^v intersect transversally (by [22]).

Now let Y and Z be subvarieties of $\mathbb{A}^N$, both invariant under scalar multiplication, and intersecting properly. Assume in addition that they intersect transversally along a dense open subset of $Y \cap Z$. Then

$$\mathrm{mult}_{\mathbf{o}}(Y \cap Z) = \mathrm{mult}_{\mathbf{o}}(Y) \cdot \mathrm{mult}_{\mathbf{o}}(Z)$$

where $\mathbf{o}$ is the origin of $\mathbb{A}^N$.

To see this, let $\mathbb{P}(Y)$, $\mathbb{P}(Z)$ be the closed subvarieties of $\mathbb{P}(\mathbb{A}^N) = \mathbb{P}^{N-1}$ associated with Y, Z. Then $\mathrm{mult}_{\mathbf{o}}(Y)$ equals the degree $\deg(\mathbb{P}(Y))$, and likewise for Z, $Y \cap Z$. Now

$$\deg(\mathbb{P}(Y \cap Z)) = \deg(\mathbb{P}(Y) \cap \mathbb{P}(Z)) = \deg(\mathbb{P}(Y)) \cdot \deg(\mathbb{P}(Z))$$

by the assumptions and the Bezout theorem (see [8], Proposition 8.4 and Example 8.1.11).

It has also been pointed out by the referee that the above alternate proof in fact holds for Richardson varieties in a minuscule G/P, since the intersections of Schubert and opposite Schubert varieties with the opposite cell are again invariant under scalar multiplication (the result analogous to Corollary 7.2.4 for a minuscule G/P follows from the results in [15]). Recall that for G a semisimple algebraic group and P a maximal parabolic subgroup of G, G/P is said to be *minuscule* if the associated fundamental weight ω of P satisfies

$$(\omega, \beta^*)\,(= 2(\omega,\beta)/(\omega,\beta))\ \leq 1$$

for all positive roots β, where (,) denotes a W-invariant inner product on $X(T)$.

7.7. **Determinantal Formula for** $\mathrm{mult}\, X_w^v$**.** In this section, we extend the Rosenthal-Zelevinsky determinantal formula (cf. [24]) for the multplicity of a Schubert variety at a T-fixed point to the case of Richardson varieties. We use the convention that the binomial coefficient $\binom{a}{b} = 0$ if $b < 0$.

Theorem 7.7.1. *(Rosenthal-Zelevinsky) Let $w = (i_1, \ldots, i_d)$ and $\tau = (\tau_1, \ldots, \tau_d)$ be such that $w \geq \tau$. Then*

$$\operatorname{mult}_\tau X_w = (-1)^{\kappa_1+\cdots+\kappa_d} \begin{vmatrix} \binom{i_1}{-\kappa_1} & \cdots & \binom{i_d}{-\kappa_d} \\ \binom{i_1}{1-\kappa_1} & \cdots & \binom{i_d}{1-\kappa_d} \\ \vdots & & \vdots \\ \binom{i_1}{d-1-\kappa_1} & \cdots & \binom{i_d}{d-1-\kappa_d} \end{vmatrix},$$

where $\kappa_q := \#\{\tau_p \mid \tau_p > i_q\}$, *for* $q = 1, \ldots, d$.

Lemma 7.7.2. $\operatorname{mult}_\tau X^v = \operatorname{mult}_{w_o\tau} X_{w_0 v}$, *where* $w_0 = (n+1-d, \ldots, n)$.

Proof. Fix a lift n_0 in $N(T)$ of w_0. The map $f : X^v \to n_0 X^v$ given by left multiplication is an isomorphism of algebraic varieties. We have $f(e_\tau) = e_{w_0\tau}$, and $n_0 X^v = n_0\overline{B^- e_v} = n_0\overline{n_0 B n_0 e_v} = \overline{B n_0 e_v} = \overline{B e_{w_0 v}} = X_{w_0 v}$. □

Theorem 7.7.3. *Let* $w = (i_1, \ldots, i_d)$, $\tau = (\tau_1, \ldots, \tau_d)$, *and* $v = (j_1, \ldots, j_d)$ *be such that* $w \geq \tau \geq v$. *Then*

$$\operatorname{mult}_\tau X_w^v = (-1)^c \left| \begin{pmatrix} \binom{i_1}{-\kappa_1} & \cdots & \binom{i_d}{-\kappa_d} \\ \binom{i_1}{1-\kappa_1} & \cdots & \binom{i_d}{1-\kappa_d} \\ \vdots & & \vdots \\ \binom{i_1}{d-1-\kappa_1} & \cdots & \binom{i_d}{d-1-\kappa_d} \end{pmatrix} \cdot \begin{pmatrix} \binom{n+1-j_d}{-\gamma_d} & \cdots & \binom{n+1-j_1}{-\gamma_1} \\ \binom{n+1-j_d}{1-\gamma_d} & \cdots & \binom{n+1-j_1}{1-\gamma_1} \\ \vdots & & \vdots \\ \binom{n+1-j_d}{d-1-\gamma_d} & \cdots & \binom{n+1-j_1}{d-1-\gamma_1} \end{pmatrix} \right|,$$

where $\kappa_q := \#\{\tau_p \mid \tau_p > i_q\}$, *for* $q = 1, \ldots, d$, *and* $\gamma_q := \#\{\tau_p \mid \tau_p < j_q\}$, *for* $q = 1, \ldots, d$, *and* $c = \kappa_1 + \cdots + \kappa_d + \gamma_1 + \cdots + \gamma_d$.

Proof. Follows immediately from Theorems 7.6.4, 7.7.1, and Lemma 7.7.2, in view of the fact that $w_0\tau = (n+1-\tau_d, \ldots, n+1-\tau_1)$ and $w_0 v = (n+1-j_d, \ldots, n+1-j_1)$. □

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