Multiplicities of Singular Points in Schubert Varieties of Grassmannians

Victor Kreiman and V. Lakshmibai

Northeastern University, Boston, MA 02115

Abstract. We give a closed-form formula for the Hilbert function of the tangent cone at the identity of a Schubert variety X in the Grassmannian in both group theoretic and combinatorial terms. We also give a formula for the multiplicity of X at the identity, and a Gröbner basis for the ideal defining $X(w) \cap O^-$ as a closed subvariety of O^-, where O^- is the opposite cell in the Grassmannian. We give conjectures for the Hilbert function and multiplicity at points other than the identity.

1 Introduction

The first formulas for the multiplicities of singular points on Schubert varieties in Grassmannians appeared in Abhyankar's results [1] on the Hilbert series of determinantal varieties (recall that a determinantal variety gets identified with the opposite cell in a suitable Schubert variety in a suitable Grassmannian). Herzog-Trung [6] generalized these formulas to give determinantal formulas for the multiplicities at the identity of all Schubert varieties in Grassmannians. Using standard monomial theory, Lakshmibai-Weyman [7] obtained a recursive formula for the multiplicities of all points in Schubert varieties in a minuscule G/P; Rosenthal-Zelevinsky [9] used this result to obtain a closed-form determinantal formula for multiplicities of all points in Grassmannians.

2 Summary of Results

Let K be the base field, which we assume to be algebraically closed, of arbitrary characteristic. Let G be $SL_n(K)$, T the subgroup of diagonal matrices in G, and B the subgroup of upper diagonal matrices in G. Let R be the root system of G relative to T, and R^+ the set of positive roots relative to B. Let W be the Weyl group of G. Note that $W = S_n$, the group of permutations of the set of n elements. Let P_d be the maximal parabolic subgroup

$$P_d = \left\{ A \in G \;\middle|\; A = \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \right\}.$$

Let R_{P_d}, $R^+_{P_d}$, and W_{P_d} denote respectively the root system, set of positive roots, and Weyl group of P_d. The quotient W/W_{P_d}, with the Bruhat order, is a distributive lattice. The map $\alpha \mapsto s_\alpha W_{P_d}$ taking a positive root to its corresponding reflection, embeds $R^+ \setminus R^+_{P_d}$ in W/W_{P_d}. We shall also denote the image by $R^+ \setminus R^+_{P_d}$. It is a sublattice of W/W_{P_d}.

A *multiset* is similar to a set, but with repetitions of entries allowed. Define the *cardinality* of a multiset S, denoted by $|S|$, to be the number of elements in S, including repetitions. Define a *uniset* to be a multiset which has no repetitions. If $\mathcal{S}$ is a set, define $\mathcal{S}^*$ to be the collection of all multisets which are made up of elements of $\mathcal{S}$.

A *chain of commuting reflections* in W/W_{P_d} is a nonempty set of pairwise-commuting reflections $\{s_{\alpha_1}, \dots, s_{\alpha_t}\}$, $\alpha_i \in R^+ \setminus R^+_{P_d}$, such that $s_{\alpha_1} > \cdots > s_{\alpha_t}$; we refer to t as the *length* of the chain. For a multiset $S \in (R^+ \setminus R^+_{P_d})^*$, define the *chainlength* of S to be the maximum length of a chain of commuting reflections in S.

Fix $w \in W/W_{P_d}$. Define S_w to be the multisets S of $(R^+ \setminus R^+_{P_d})^*$, such that the product of every chain of commuting reflections in S is less than or equal to w; similarly, define S'_w to be the unisets of $(R^+ \setminus R^+_{P_d})^*$ having the same property. For m a positive integer, define

$$S_w(m) = \{S \in S_w \,:\, |S| = m\}$$

$$S'_w(m) = \{S \in S'_w \,:\, |S| = m\}.$$

We can now state our two main results. First, letting $X(w)$ denote the Schubert variety of G/P_d corresponding to $w \in W/W_{P_d}$, the Hilbert function of the tangent cone to $X(w)$ at the identity is given by

Theorem 1 $h_{TC_{id}X(w)}(m) = |S_w(m)|, m \in \mathbb{N}$.

Second, letting M denote the maximum cardinality of any element of S'_w, the multiplicity at the identity is given by

Theorem 2 $mult_{id}X(w) = |\{S \in S'_w \,:\, |S| = M\}|$.

3 Preliminaries

3.1 Multiplicity of an Algebraic Variety at a Point

Let B be a graded, affine K-algebra such that B_1 generates B (as a K-algebra). Let $X = \mathrm{Proj}(B)$. The function $h_B(m)$ (or $h_X(m)$) $= \dim_K B_m$, $m \in \mathbb{Z}$ is called the *Hilbert function* of B (or X). There exists a polynomial $P_B(x)$ (or $P_X(x)$) $\in \mathbb{Q}[x]$, called the *Hilbert polynomial* of B (or X), such that $f_B(m) = P_B(m)$ for $m \gg 0$. Let r denote the degree of $P_B(x)$. Then $r = \dim(X)$, and the leading coefficient of $P_B(x)$ is of the form $c_B/r!$, where $c_B \in \mathbb{N}$. The integer c_B is called the *degree of* X, and denoted $\deg(X)$. In the sequel we shall also denote $\deg(X)$ by $\deg(B)$.

Let X be an algebraic variety, and let $P \in X$. Let $A = \mathcal{O}_{X,P}$ be the stalk at P and $\mathfrak{m}$ the unique maximal ideal of the local ring A. Then the *tangent cone* to X at P, denoted $\mathrm{TC}_P(X)$, is defined to be $\mathrm{Spec}(\mathrm{gr}(A, \mathfrak{m}))$, where $\mathrm{gr}(A, \mathfrak{m}) = \oplus_{j=0}^{\infty} \mathfrak{m}^j / \mathfrak{m}^{j+1}$. The *multiplicity* of X at P, denoted $\mathrm{mult}_P(X)$, is defined to be $\deg(\mathrm{Proj}(\mathrm{gr}(A, \mathfrak{m})))$. If $X \subset K^n$ is an affine closed subvariety, and $m_P \subset K[X]$ is the maximal ideal corresponding to $P \in X$, then $\mathrm{gr}(K[X], m_P) = \mathrm{gr}(A, \mathfrak{m})$.

3.2 Monomial Orders, Gröbner Bases, and Flat Deformations

Let A be the polynomial ring $K[x_1, \cdots, x_n]$. A *monomial order* $\succ$ on the set of monomials in A is a total order such that given monomials m, m_1, m_2, $m \neq 1$, $m_1 \succ m_2$, we have $mm_1 \succ m_1$ and $mm_1 \succ mm_2$. The largest monomial (with respect to $\succ$) present in a polynomial $f \in A$ is called the *initial term* of f, and is denoted by $\mathrm{in}(f)$.

The *lexicographic order* is a total order defined in the following manner. Assume the variables $x_1, \ldots, x_n$ are ordered by $x_n > \cdots > x_1$. A monomial m of degree r in the polynomial ring A will be written in the form $m = x_{i_1} \cdots x_{i_r}$, with $n \geq i_1 \geq \cdots \geq i_r \geq 1$. Then $x_{i_1} \cdots x_{i_r} \succ x_{j_1} \cdots x_{j_s}$ in the lexicographic order if and only if either $r > s$, or $r = s$ and there exists an $l < r$ such that $i_1 = j_1, \ldots, i_l = j_l, i_{l+1} > j_{l+1}$. It is easy to check that the lexicographic order is a monomial order.

Given an ideal $I \subset A$, denote by $\mathrm{in}(I)$ the ideal generated by the initial terms of the elements in I. A finite set $\mathcal{G} \subset I$ is called a *Gröbner basis* of I (with respect to the monomial order $\succ$), if $\mathrm{in}(I)$ is generated by the initial terms of the elements of $\mathcal{G}$.

Flat Deformations: Given a monomial order and an ideal $I \subset A$, there exists a flat family over $\mathrm{Spec}(K[t])$ whose special fiber ($t = 0$) is $\mathrm{Spec}(A/\mathrm{in}(I))$ and whose generic fiber (t invertible) is $\mathrm{Spec}(A/I \otimes K[t, t^{-1}])$. Further, if I is homogeneous, then the special fiber and generic fiber have the same Hilbert function (see [4] for details).

3.3 Grassmannian and Schubert Varieties

The Plücker Embedding: Let d be such that $1 \leq d < n$. The *Grassmannian* $G_{d,n}$ is the set of all $d-$dimensional subspaces $U \subset K^n$. Let U be an element of $G_{d,n}$ and $\{a_1, \ldots a_d\}$ a basis of U, where each a_j is a vector of the form

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}, \text{ with } a_{ij} \in K.$$

Thus, the basis $\{a_1, \cdots, a_d\}$ gives rise to an $n \times d$ matrix $A = (a_{ij})$ of rank d, whose columns are the vectors $a_1, \cdots, a_d$.

We have a canonical embedding

$$p : G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d K^n) \ , \ U \mapsto [a_1 \wedge \cdots \wedge a_d]$$

called the *Plücker embedding.* Let

$$I_{d,n} = \{\underline{i} = (i_1, \ldots, i_d) \in \mathbb{N}^d \ : 1 \leq i_1 < \cdots < i_d \leq n\} \ .$$

Then the projective coordinates (*Plücker coordinates*) of points in $\mathbb{P}(\wedge^d K^n)$ may be indexed by $I_{d,n}$; for $\underline{i} \in I_{d,n}$, we shall denote the $\underline{i}$-th component of p by $p_{\underline{i}}$, or $p_{i_1,\cdots,i_d}$. If a point U in $G_{d,n}$ is represented by the $n \times d$ matrix A as above, then $p_{i_1,\cdots,i_d}(U) = \det(A_{i_1,\ldots,i_d})$, where $A_{i_1,\ldots,i_d}$ denotes the $d \times d$ submatrix whose rows are the rows of A with indices $i_1, \ldots, i_d$, in this order.

Identification of G/P_d with $G_{d,n}$: Let G, T, B, and P_d be as in Section 2. Let $\{e_1, \ldots, e_n\}$ be the standard basis for K^n. For the natural action of G on $\mathbb{P}(\wedge^d K^n)$, the isotropy group at $[e_1 \wedge \cdots \wedge e_d]$ is P_d, while the orbit through $[e_1 \wedge \cdots \wedge e_d]$ is $G_{d,n}$. Thus we obtain an identification of G/P_d with $G_{d,n}$. We also note that W/W_{P_d} $(= S_n/(S_d \times S_{n-d}))$ may be identified with $I_{d,n}$.

Schubert Varieties: For the action of G on $G_{d,n}$, the T-fixed points are precisely $\{[e_{\underline{i}}], \underline{i} \in I_{d,n}\}$, where $e_{\underline{i}} = e_{i_1} \wedge \cdots \wedge e_{i_d}$. The *Schubert variety* $X_{\underline{i}}$ associated to $\underline{i}$ is the Zariski closure of the B-orbit $B[e_{\underline{i}}]$ with the canonical reduced scheme structure.

We have a bijection between {Schubert varieties in $G_{d,n}$} and $I_{d,n}$. The partial order on Schubert varieties given by inclusion induces a partial order (called the *Bruhat order*) on $I_{d,n}$ $(= W/W_{P_d})$; namely, given $\underline{i} = (i_1, \ldots, i_d)$, $\underline{j} = (j_1, \ldots, j_d) \in I_{d,n}$,

$$\underline{i} \geq \underline{j} \iff i_t \geq j_t, \text{for all } 1 \leq t \leq d.$$

We note the following facts for Schubert varieties in the Grassmannian (see [5] or [8] for example):

- **Bruhat Decomposition**: $X_{\underline{i}} = \bigcup\limits_{\underline{j} \leq \underline{i}} B[e_{\underline{j}}]$.
- **Dimension**: $\dim X_{\underline{i}} = \sum\limits_{1 \leq t \leq d} i_t - t$.
- **Vanishing Property of a Plücker Coordinate**:

$$p_{\underline{j}}\big|_{X_{\underline{i}}} \neq 0 \iff \underline{i} \geq \underline{j}.$$

Standard Monomials: A monomial $f = p_{\theta_1} \cdots p_{\theta_t}$, $\theta_i \in W/W_{P_d}$ is said to be *standard* if

$$\theta_1 \geq \cdots \geq \theta_t. \tag{1}$$

Such a monomial is said to be *standard on the Schubert variety* $X(\theta)$, if in addition to (1), we have $\theta \geq \theta_1$.

Let $w \in W/W_{P_d}$. Let $R(w) = K[X(w)]$, the homogeneous coordinate ring for $X(w)$, for the Plücker embedding. Recall the following two results from standard monomial theory (cf. [5]).

Theorem 3 *The set of standard monomials on $X(w)$ of degree m is a basis for $R(w)_m$.*

Theorem 4 *For $w \in W/W_{P_d}$, let I_w be the ideal in $K[G_{d,n}]$ generated by $\{p_\theta, \theta \not\leq w\}$. Then $R(w) = K[G_{d,n}]/I_w$.*

The Opposite Big Cell O^-: Let U^- denote the unipotent lower triangular matrices of $G = SL_n(K)$. Under the canonical projection $G \to G/P_d$, $g \mapsto gP_d\,(= g[e_{id}])$, U^- maps isomorphically onto its image $U^-[e_{id}]$. The set $U^-[e_{id}]$ is called the *opposite big cell* in $G_{d,n}$, and is denoted by O^-. Thus, O^- may be identified with

$$\left\{ \begin{pmatrix} & \mathrm{Id}_{d\times d} & \\ x_{d+1\,1} & \cdots & x_{d+1\,d} \\ \vdots & & \vdots \\ x_{n\,1} & \cdots & x_{n\,d} \end{pmatrix}, \quad x_{ij} \in K, \quad d+1 \leq i \leq n, 1 \leq j \leq d \right\}. \tag{2}$$

Thus we see that O^- is an affine space of dimension $(n-d) \times d$, with id as the origin; further $K[O^-]$ can be identified with the polynomial algebra $K[x_{-\beta}, \beta \in R^+ \setminus R^+_{P_d}]$. To be very precise, denoting the elements of R as in [2], we have $R^+ \setminus R^+_{P_d} = \{\epsilon_j - \epsilon_i,\ d+1 \leq i \leq n,\ 1 \leq j \leq d\}$; given $\beta \in R^+ \setminus R^+_{P_d}$, say $\beta = \epsilon_j - \epsilon_i$, we identify $x_{-\beta}$ with x_{ij}. We denote by $s_{(i,j)}$ (or $s_{(j,i)}$) the reflection corresponding to β, namely, the transposition switching i and j.

Evaluation of Plücker Coordinates on O^-: Let $\underline{j} \in I_{d,n}$. We shall denote the Plücker coordinate $p_{\underline{j}}|_{O^-}$ by $f_{\underline{j}}$. Let us denote a typical element $A \in O^-$ by $\begin{pmatrix} \mathrm{Id}_{d\times d} \\ X \end{pmatrix}$. Then $f_{\underline{j}}$ is simply a minor of X as follows. Let $\underline{j} = (j_1, \ldots, j_d)$, and let j_r be the largest entry $\leq d$. Let $\{k_1, \ldots, k_{d-r}\}$ be the complement of $\{j_1, \ldots, j_r\}$ in $\{1, \ldots, d\}$. Then this minor of X is given by column indices $k_1, \ldots, k_{d-r}$ and row indices $j_{r+1}, \ldots, j_d$ (here the rows of X are indexed as $d+1, \ldots, n$).

Conversely, given a minor of X, say, with column indices $b_1, \ldots, b_s$, and row indices $i_{d-s+1}, \ldots, i_d$, then that minor is the evaluation of $f_{\underline{j}}$ at X, where $\underline{j} = (j_1, \ldots, j_d)$ may be described as follows: $\{j_1, \ldots, j_{d-s}\}$ is the complement of $\{b_1, \ldots, b_s\}$ in $\{1, \ldots, d\}$, and $j_{d-s+1}, \ldots, j_d$ are simply the row indices (again, the rows of X are indexed as $d+1, \ldots, n$).

Note that if $\underline{j} = (1, \ldots, d)$, then $p_{\underline{j}}$ evaluated at X is 1. In the above discussion, therefore, we must consider the element 1 (in $K[O^-]$) as the minor of X with row indices (and column indices) given by the empty set.

Example 1 *Consider $G_{2,4}$. Then*

$$O^- = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix}, \; x_{ij} \in K \right\}.$$

On O^-, we have $p_{12} = 1$, $p_{13} = x_{32}$, $p_{14} = x_{42}$, $p_{23} = x_{31}$, $p_{24} = x_{41}$, $p_{34} = x_{31}x_{42} - x_{41}x_{32}$.

Note that each of the Plücker coordinates is homogeneous in the local coordinates x_{ij}.

4 The Hilbert Function of $\mathrm{TC}_{id}X(w)$

In view of the Bruhat decomposition, in order to determine the multiplicity at a singular point x, it is enough to determine the multiplicity of the T-fixed point in the B orbit Bx. In this section, we shall discuss the behavior at a particular T-fixed point, namely the identity.

4.1 The Variety Y(w)

We define $Y(w) \subset G_{d,n}$ to be $X(w) \cap O^-$. Since $Y(w) \subset X(w)$ is open dense, and $id \in Y(w)$, we have that $\mathrm{TC}_{id}Y(w) = \mathrm{TC}_{id}X(w)$. As a consequence of Theorem 4, $Y(w) \subset O^-$ is defined as an algebraic subvariety by the homogeneous polynomials $f_\theta, \theta \not\leq w$; further, $id \in O^-$ corresponds to the origin. Thus we have that $\mathrm{gr}(K[Y(w)], m_{id}) = K[Y(w)]$. Hence,

$$\begin{aligned} \mathrm{TC}_{id}X(w) = \mathrm{TC}_{id}Y(w) &= \mathrm{Spec}(\mathrm{gr}(K[Y(w)], m_{id})) \\ &= \mathrm{Spec}(K[Y(w)]) = Y(w). \end{aligned} \tag{3}$$

4.2 Monomials and Multisets

For a monomial $p = x_{\alpha_{i_1}} \cdots x_{\alpha_{i_m}} \in K[O^-]$, define *Multisupp*$(p)$ to be the multiset $\{\alpha_{i_1}, \ldots \alpha_{i_m}\}$. It follows immediately from the definition that Multisupp gives a bijection between the monomials of $K[O^-]$ and the multisets of $(R^+ \setminus R^+_{P_d})^*$, pairing the square-free monomials with the unisets. Let $w \in W/W_{P_d}$. We call a monomial *w-good* if it maps under Multisupp to an element of S_w. Note that the w-good square-free monomials are precisely those which map to S'_w.

Define a monomial order $\succ$ on $K[O^-]$ in the following manner. We say $x_{i,j} > x_{i',j'}$ if $i > i'$, or if $i = i'$ and $j < j'$. Note that this extends the partial order $x_\alpha > x_\beta \iff s_\alpha > s_\beta$ (in the Bruhat order). The monomials are then ordered using the lexicographic order.

Define the monomial ideal $J_w \subset K[O^-]$ to be the ideal generated by $\{\text{in} f_\theta, \theta \not\leq w\}$, and let $A_w = K[O^-]/J_w$. With our ordering, Multisupp$(\text{in} f_\theta)$ is a commuting chain of reflections whose product is θ. Thus the non w-good monomials form a vector space basis for J_w, and therefore the w-good monomials form a basis for A_w.

4.3 Sketch of Proof of Theorems 1 and 2

In view of (3) and the above discussion, Theorem 1 follows immediately from

Lemma 1 $h_{K[Y(w)]}(m) = h_{A_w}(m)\,, m \in \mathbb{N}.$

Theorem 2 is also a consequence. Indeed,

$$\text{mult}_{id} X(w) = \deg(K[\text{TC}_{id} X(w)]) = \deg(K[Y(w)]) = \deg(A_w).$$

Since A_w is an affine quotient of an ideal generated by square-free monomials, letting M be the maximum degree of a square-free monomial in A_w, we have (cf. [3])

$$\begin{aligned}\deg(A_w) &= |\{p \in A_w \,:\, p \text{ is a square-free monomial and } \deg(p) = M\}| \\ &= |\{p \in K[O^-] \,:\, p \text{ is a square-free w-good monomial and } \deg(p) = M\}| \\ &= |\{S \in S'_w \,:\, |S| = M\}|,\end{aligned}$$

yielding Theorem 2.

The proof of Lemma 1 relies on an inductive argument which shows directly that both functions agree for all positive integers m. Note that $K[Y(w)] = K[X(w)]_{(p_{id})}$. Thus, as a consequence of Theorem 3, $K[Y(w)]$ has a basis consisting of monomials of the form $f_{\theta_1} \cdots f_{\theta_t}, w \geq \theta_1 \geq \cdots \geq \theta_t$. If $SM_w(m)$ denotes the basis elements of degree m, then $h_{Y(w)}(m) = |SM_w(m)|$. Letting $d = d_w$ be the degree of w (see section 4.4 below for definition), as a consequence of standard monomial theory we have

$$SM_w(m+d) = SM_w(m) \,\dot{\cup}\, SM_H(m+d) \tag{4}$$

where $SM_H(m+d) = \bigcup_{w_i} SM_{w_i}(m+d)$, the union being taken over the divisors $X(w_i)$ of $X(w)$ (cf. [7]).

We have that $|SM_H(m+d)| = |\bigcup_{w_i} SM_{w_i}(m+d)|$ can be set-theoretically written as the integral linear combination of terms of the form $|SM_{w_i}(m+d)|$

and terms of the form $|SM_{w_j}(m+d) \cap \cdots \cap SM_{w_k}(m+d)|$. Further, it can be shown that

$$SM_{w_j}(m+d) \cap \cdots \cap SM_{w_k}(m+d) = SM_\theta(m+d),$$

where θ is given by $X(\theta) = X(w_j) \cap \cdots \cap X(w_k)$. (Note that $I_{d,n}$ being a distributive lattice implies that for $\tau, \phi \in I_{d,n}$, $X(\tau) \cap X(\phi)$ is irreducible.) Thus,

$$|SM_H(m+d)| = \sum_{w'<w} a_{w'} |SM_{w'}(m+d)|, \text{ for some } a_{w'} \in \mathbb{Z}. \tag{5}$$

Taking cardinalities of both sides of (4), we obtain

$$h_{K[Y(w)]}(m+d) = h_{K[Y(w)]}(m) + \sum_{w'<w} a_{w'} h_{K[Y(w')]}(m+d).$$

Equivalently, $h_{K[Y(w)]}$ satisfies the difference equation

$$\phi(w, m+d) = \phi(w, m) + \sum_{w'<w} a_{w'} \phi(w', m+d). \tag{6}$$

To prove Lemma 1, it suffices to show that $h_{A_w}(m)$ satisfies (6) for all $m \in \mathbb{Z}_{\geq 0}$, since it is a straightforward verification that $h_{K[Y(w)]}(m)$ and $h_{A_w}(m)$ have the same initial conditions.

As stated earlier, $K[A_w]$ has as basis the w-good monomials of $K[O^-]$, which are in bijection with the elements of S_w. Thus $h_{K[A_w]}(m) = |S_w(m)|$, and it suffices to show that $|S_w(m)|$ satisfies (6). We can write

$$S_w(m+d) = (S_w(m+d) \setminus S_H(m+d)) \dot\cup S_H(m+d), \tag{7}$$

where $S_H(m+d) = \bigcup_{w_i} S_{w_i}(m+d)$, the union being over the divisors $X(w_i)$ of $X(w)$. Following the identical arguments used to deduce (5) (replacing "SM" by "S" everywhere), one obtains

$$|S_H(m+d)| = \sum_{w'<w} a_{w'} |S_{w'}(m+d)|, \tag{8}$$

for the same integers $a_{w'}$ as in (5).

Establishing an explicit bijection between $S_w(m+d) \setminus S_H(m+d)$ and $S_w(m)$ completes the proof, for then (taking cardinalities of both sides of (7)), one sees that $h_{A_w}(m)$ satisfies (6) for all $m \in \mathbb{Z}_{\geq 0}$.

In view of the discussion of flat deformations in Section 3.2, Lemma 1 also implies

Corollary 1 *The set $\{f_\theta, \theta \not\leq w\} \subset K[O^-]$ forms a Gröbner basis for the ideal it generates.*

4.4 Combinatorial Interpretation

We call a multiset S of $(R^+ \setminus R^+_{P_d})^*$ a *t-multipath*, if the chainlength of S is t. If S has no repeated elements (i.e. it is a uniset), then we call it a *t-unipath*. Define $s \in S$ to be a *chain-maximal element of* S if there is no element in S strictly greater than s which commutes with s. Any t-multipath S can be written in the following manner as the union of t nonintersecting 1-multipaths: if S_i is the i^{th} 1-multipath, then S_{i+1} is the multiset of chain-maximal elements (including repetitions) of $S \setminus \cup_{k=1}^{i} S_k$ (for $i = 0, \cdots, t-1$, where S_0 is defined to be the empty set). If the t-multipath S is a t-unipath, then each S_i will be a 1-unipath.

Fix $w \in W/W_{P_d}$. There is a unique expression $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_{d_w}}}$ such that $s_{\alpha_{i_k}} > s_{\alpha_{i_{k+1}}}$ for all k, and all the reflections pairwise commute; d_w is called the *degree* of w.

Example 2 *Let* $w = (3,5,7,8) \in I_{4,8}$. *Then* $w = s_{(8,1)}\, s_{(7,2)}\, s_{(5,4)}$, *where* $s_{(8,1)} > s_{(7,2)} > s_{(5,4)}$ *is a chain of commuting reflections. Thus* $d_w = 3$.

Let $H_j = \{\alpha \in R^+ \setminus R^+_{P_d} \mid s_\alpha \le s_{\alpha_{i_j}}\}$. We say that a t-multipath S is w-good if, when written as the union of weighted 1-multipaths $\cup_{k=1}^{t} S_k$ as above, we have that the elements of S_j are in H_j, $j = 1, \cdots, t$. Any multiset in $(R^+ \setminus R^+_{P_d})^*$ is a t-multipath for some t; it is said to be w-good if the corresponding t-multipath is w-good.

It can be seen that the combinatorial property that a multiset (resp. uniset) S of $(R^+ \setminus R^+_{P_d})^*$ is w-good is equivalent to the group-theoretic property that $S \in S_w$ (resp. $S \in S'_w$). Thus Theorem 1 is equivalent to the assertion that $h_{\mathrm{TC}_{id}X(w)}(m)$ is the number of w-good multisets of $(R^+ \setminus R^+_{P_d})^*$ of degree m. Letting M be the maximum cardinality of a w-good uniset, Theorem 2 is equivalent to the assertion that $\mathrm{mult}_{id}X(w)$ is the number of w-good unisets of cardinality M.

Example 3 *Let* $w = s_{(15,2)}\, s_{(13,4)}\, s_{(10,5)} \in I_{7,16}$. *We have that* $s_{(15,2)} > s_{(13,4)} > s_{(10,5)}$ *is a chain of commuting reflections, and thus* $d_w = 3$.

The diagram below shows the lattice $R^+ \setminus R^+_{P_7}$, *where the reflection* $s_{(i,j)}$ *is denoted by* i, j. *The set* S *of reflections which lie along the three broken-line paths is an example of a* w*-good uniset of maximum cardinality. In fact, any* w*-good uniset of maximum cardinality can be seen as the set of reflections lying on three paths in the lattice, satisfying the following properties:*

- *One path starts and ends at "X", the second at "Y", and the third at "Z".*
- *Each path can move only down or to the right.*
- *The paths do not intersect.*

Thus the number of ways of drawing three such paths is $\mathit{mult}_{id}X(w)$.

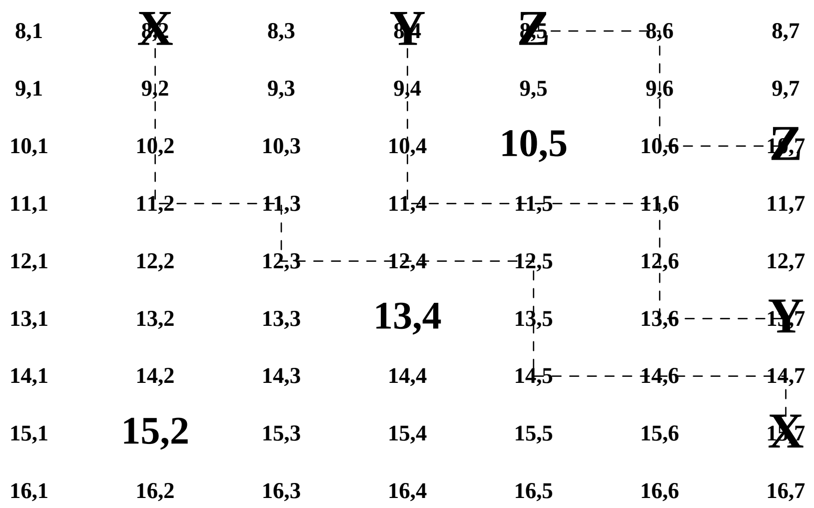

Fig. 1.

5 Conjectures on the Behavior at Other Points

Let $w, \tau \in W/W_{P_d}$. Define $S_{w,\tau}$ to be the multisets S of $(R^+ \setminus R^+_{P_d})^*$, such that for every chain of commuting reflections $s_{\alpha_1} > \cdots > s_{\alpha_t},\ s_{\alpha_i} \in S$, we have that $w \geq \tau s_{\alpha_1} \cdots s_{\alpha_t}$; define $S'_{w,\tau}$ to be the unisets of $(R^+ \setminus R^+_{P_d})^*$ having the same property. For m a positive integer, define

$$S_{w,\tau}(m) = \{S \in S_{w,\tau} \ : \ |S| = m\}$$

$$S'_{w,\tau}(m) = \{S \in S'_{w,\tau} \ : \ |S| = m\}.$$

We state two conjectures. First, the Hilbert function $h_{\mathrm{TC}_\tau X(w)}(m)$ of the tangent cone to $X(w)$ at τ is given by

Conjecture 1 $h_{TC_\tau X(w)}(m) = |S_{w,\tau}(m)|, m \in \mathbb{N}$.

Second, letting M denote the maximum cardinality of an element of $S'_{w,\tau}$, the multiplicity $\mathrm{mult}_\tau X(w)$ of $X(w)$ at τ is given by

Conjecture 2 $mult_\tau X(w) = |\{S \in S'_{w,\tau} \ : \ |S| = M\}|$.

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