EQUIVARIANT LITTLEWOOD-RICHARDSON SKEW TABLEAUX

VICTOR KREIMAN

Abstract. We give a positive equivariant Littlewood-Richardson rule also discovered independently by Molev. Our proof generalizes a proof by Stembridge of the classical Littlewood-Richardson rule. We describe a weight-preserving bijection between our indexing tableaux and trapezoid puzzles which restricts to a bijection between positive indexing tableaux and Knutson-Tao puzzles.

Contents

1. Introduction

In [MS], Molev and Sagan introduced a rule in terms of barred tableaux for computing the structure constants $c^{\nu}_{\lambda,\mu}$ for products of two factorial Schur functions. Knutson and Tao [KT] realized that under a suitable specialization these are the structure constants $C^{\nu}_{\lambda,\mu}$ for products of two Schubert classes in the equivariant cohomology ring of the Grassmannian. Knutson and Tao [KT] also gave a new rule for computing $C^{\nu}_{\lambda,\mu}$, i.e., an equivariant Littlewood-Richardson rule, which is manifestly positive in the sense of Graham [Gr]. Their rule was expressed in terms of puzzles, generalizations of combinatorial objects first introduced by Knutson, Tao, and Woodward [KTW].

We describe a new nonnegative equivariant Littlewood-Richardson rule, expressed in terms of skew barred tableaux, which was also discovered independently by Molev [Mo1]. By nonnegative we mean that all of the coefficients are either positive or zero; restricting to the positive coefficients then yields a positive rule. The rule includes several equivalent combinatorial tests for determining in advance

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which skew barred tableaux result in positive coefficients. Although our rule is similar to the Molev-Sagan rule [MS], it produces a different expression for $c^{\nu}_{\lambda,\mu}$ (see Examples 2.11 and 10.3). For example, the Molev-Sagan rule is not manifestly positive. We remark that unlike the Knutson-Tao rule, whose positivity is obvious from its statement, the nonnegativity and positivity of our rule require proof.

In this paper, we compute the structure constants $c^{\nu}_{\lambda,\mu}$ (as do both [MS] and [Mo1]), and then determine the structure constants $C^{\nu}_{\lambda,\mu}$ by specialization (as does [Mo1]). Our strategy for proving our rule for the structure constants $c^{\nu}_{\lambda,\mu}$ is to generalize a concise proof by Stembridge [St] of a standard Littlewood-Richardson rule from Schur functions to factorial Schur functions; a similar method is used by [Kr2]. This method in fact yields a more general result, namely, a generalization of Zelevinsky's extension of the Littlewood-Richardson rule [Z].

We illustrate a weight-preserving bijection Φ between skew barred tableaux and trapezoid puzzles, combinatorial objects generalizing Knutson-Tao puzzles. The bijection Φ restricts to a bijection between the skew barred tableaux indexing positive coefficients and Knutson-Tao puzzles. This gives a new proof of Knutson and Tao's equivariant Littlewood-Richardson rule, and also demonstrates that our positive rule is really the same rule as Knutson and Tao's, just expressed in terms of different combinatorial indexing sets. Our representation of the bijections generalizes Tao's 'proof without words' [V, Figure 11], which gives a bijection between tableaux and puzzles in the nonequivariant setting.

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2. Statement of Results

Let $\mathbb{N}$ denote the set of nonnegative integers, and let $n \geq d$ be fixed positive integers. For $m \in \mathbb{N}$, define $m' := d + 1 - m$. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{N}^d$, define $|\lambda| = \lambda_1 + \cdots + \lambda_d$. Denote by $\mathcal{P}_d$ the set of all such λ which are **partitions**, i.e., such that $\lambda_1 \geq \cdots \geq \lambda_d$, and by $\mathcal{P}_{d,n}$ the set of all such partitions for which $\lambda_1 \leq n - d$. Let $\lambda = (\lambda_1, \ldots, \lambda_d), \mu = (\mu_1, \ldots, \mu_d), \rho = (d-1, d-2, \ldots, 0)$, and $\mathbf{1} = (1, \ldots, 1)$ be fixed elements of $\mathcal{P}_d$. For any sequence $i = i_1, i_2, \ldots, i_t,\ i_j \in \{1, \ldots, d\}$, define the **content of** i to be $\omega(i) = (\xi_1, \ldots, \xi_d) \in \mathbb{N}^d$, where ξ_k is the number of k's in the sequence.

2.1. Defining the Structure Constants $c^{\nu}_{\lambda,\mu}$ for Products of Factorial Schur Functions.

A **reverse Young diagram** is a right and bottom justified array of boxes. To μ we associate the reverse Young diagram whose bottom row has length μ_1, next to bottom row has length μ_2, etc. We also denote this reverse Young diagram by μ. The columns of a reverse Young diagram are numbered from right to left and the rows from bottom to top.

A **reverse tableau of shape** μ is a filling of each box of μ with an integer in $\{1, \ldots, d\}$ in such a way that the entries weakly increase along any row from left to right and strictly increase along any column from top to bottom. Let $\mathcal{R}(\mu)$ denote the set of all reverse tableaux of shape μ. Let $x_1, \ldots, x_d$ be a finite set of variables

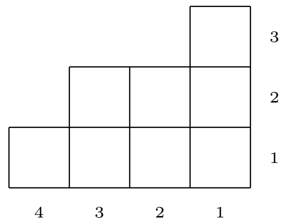


FIGURE 1. The reverse Young diagram $(4,3,1)$, with rows and columns numbered.

and $(y_i)_{i\in\mathbb{N}_{>0}}$ an infinite set of variables. For $R \in \mathcal{R}(\mu)$, define

$$(x\,|\,y)^R = \prod_{a\in R} \left(x_a - y_{a'+c(a)-r(a)}\right),$$

where for entry $a \in R$, $c(a)$ and $r(a)$ are the column and row numbers of a respectively. The **factorial Schur polynomial** is defined to be

$$s_\mu(x\,|\,y) = \sum_{R\in\mathcal{R}(\mu)} (x\,|\,y)^R.$$

The factorial Schur function is usually expressed in the literature in terms of Young tableaux rather than reverse tableaux; we show the equivalence of the two formulations below. Factorial Schur functions are special cases of Lascoux and Schützenberger's double Schubert polynomials [LS1, LS2]. The factorial Schur function $s_\mu(x\,|\,y)$, under a certain specialization of the y variables, was first defined by Biedenbarn and Louck [BL1, BL2], and further studied by Chen and Louck [CL]. The more general factorial Schur function $s_\mu(x\,|\,y)$ is due to Macdonald [Ma2] and Goulden and Greene [GG]. Factorial Schur functions appear in the study of the center of the enveloping algebra $U(\mathfrak{gl}_n)$ (see Okounkov [Ok], Okounkov and Olshanski [OO], Nazarov [Na], Molev [Mo2, Mo1], and Molev and Sagan [MS]).

We check that our definition of factorial Schur function agrees with the version appearing in [Ma2] and [MS], which is expressed in terms of Young tableaux with entries in $\{1,\dots,d\}$. Replacing each entry a in a reverse tableau R by a' and rotating the resulting tableau by 180 degrees, one obtains a Young tableau T. This operation defines a bijection between reverse tableax of shape μ and Young tableaux of shape μ with entries in $\{1,\dots,d\}$. The polynomials $(x\,|\,y)^T$, as defined in [MS], and $(x\,|\,y)^R$, as defined above, are related by a fixed permutation on the indices of the x_i's, namely the involution $i \mapsto i'$. Thus the equivalence of the two definitions follows from the fact that factorial Schur functions are symmetric in the x_i's. (Corollary 5.4 also establishes the equivalence of the two definitions.)

From the definition of $s_\mu(x\,|\,y)$, one sees that

$$s_\mu(x\,|\,y) = s_\mu(x) + \text{ terms of lower degree in the } x_i\text{'s},$$

where $s_\mu(x)$ is the Schur function in $x_1,\dots,x_d$. Since the Schur functions form a $\mathbb{Z}$-basis for $\mathbb{Z}[x_1,\dots,x_d]^{S_d}$, the factorial Schur functions must form a $\mathbb{Z}[y]$-basis for $\mathbb{Z}[y][x_1,\dots,x_d]^{S_d}$. Thus

$$s_\lambda(x\,|\,y)s_\mu(x\,|\,y) = \sum c^\nu_{\lambda,\mu} s_\nu(x\,|\,y), \tag{1}$$

for some polynomials $c^\nu_{\lambda,\mu} \in \mathbb{Z}[y]$, where the summation is over all $\nu \in \mathcal{P}_d$.

We write $\mu \subseteq \nu$ if $\mu_i \leq \nu_i$, $i = 1, \ldots, d$. Using a vanishing theorem of Okounkov [Ok], Molev and Sagan prove [MS, Theorem 3.1]

$$\mu \not\subseteq \nu \implies c_{\lambda,\mu}^{\nu} = 0. \tag{2}$$

From the definition one sees that $s_\mu(x \,|\, y)$ is a homogeneous polynomial of degree $|\mu|$. Therefore if $c_{\lambda,\mu}^{\nu} \neq 0$, then $|\lambda| + |\mu| - |\nu| = \deg(c_{\lambda,\mu}^{\nu})$. If $|\lambda| + |\mu| - |\nu| = 0$, then $c_{\lambda,\mu}^{\nu} \in \mathbb{Z}$ is the classical Littlewood-Richardson coefficient (see [F1], [LR], [Sa]).

2.2. **Computing the Structure Constants $c_{\lambda,\mu}^{\nu}$.** The **skew diagram** $\lambda * \mu$ is obtained by placing the Young diagram λ above and to the right of the reverse Young diagram μ (see Figure 2). A **skew barred tableau L of shape** $\lambda * \mu$ is a filling of each box of the subdiagram λ of $\lambda * \mu$ with an element of $\{1, \ldots, d\}$ and each box of the subdiagram μ of $\lambda * \mu$ with an element of $\{1, \ldots, d\} \cup \{\overline{1}, \ldots, \overline{d}\}$, in such a way that the values of the entries, without regard to whether or not they are barred, weakly increase along any row from left to right and strictly increase along any column from top to bottom. The **unbarred column word of** L, denoted by L^u, is the sequence of unbarred entries of L beginning at the top of the rightmost column, reading down, then moving to the top of the next to rightmost column and reading down, etc. (the barred entries are just skipped over in this process). We say that that the unbarred column word of L is **Yamanouchi** if, when one writes down the word and stops at any point, one will have written at least as many ones as twos, at least as many twos as threes, ..., at least as many $(d-1)$'s as d's. The **unbarred content of** L is $\omega(L^u)$, the content of the unbarred column word.

Definition 2.3. *An* ***equivariant Littlewood-Richardson skew tableau*** *is a skew barred tableau whose unbarred column word is Yamanouchi. We denote the set of all equivariant Littlewood-Richardson skew tableaux of shape $\lambda * \mu$ and unbarred content ν by $\mathcal{LR}_{\lambda,\mu}^{\nu}$.*

We remark that this definition forces the i-th row of λ to consist of λ_i unbarred i's.

For L a skew barred tableau and $a \in L$, denote by $L_{<a}^u$ the portion of the unbarred column word of L which comes before reaching a when reading entries from L. Define

$$c_L = \prod_{\substack{a \in L \\ a \text{ barred}}} \left(y_{|a|' + \omega(L_{<a}^u)_{|a|}} - y_{|a|' + c(a) - r(a)} \right), \tag{3}$$

where $r(a)$ and $c(a)$ are the row and column numbers of a considered as entries of μ (see Figure 1), and $|a|' = d + 1 - |a|$ (we use the absolute value symbol, $|a|$, to stress that we are interested in the integer *value* of the barred entry a). As usual, the trivial product is defined to be 1. The main result of this paper, which is proven in Sections 5, 7, and 8, is the following

Theorem 2.4. $c_{\lambda,\mu}^{\nu} = \sum_{L \in \mathcal{LR}_{\lambda,\mu}^{\nu}} c_L.$

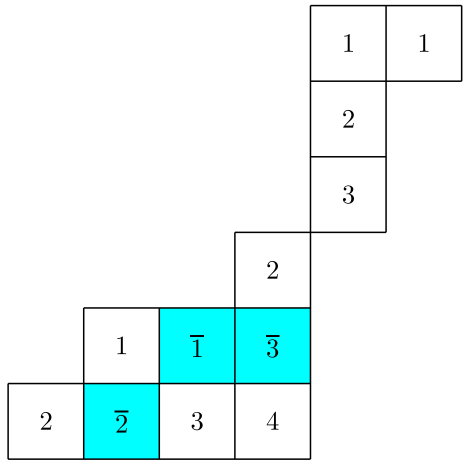


FIGURE 2. An equivariant Littlewood-Richardson skew tableau of shape $\lambda * \mu$ and unbarred content μ, where $\lambda = (2,1,1)$, $\mu = (4,3,1)$, and $\nu = (3,3,2,1)$. The unbarred column word, $1,1,2,3,2,4,3,1,2$, is Yamanouchi, as required.

Example 2.5. *Let L be the equivariant Littlewood-Richardson skew tableau of Figure 2. Suppose that $d = 4$. Consider the entry $a = \overline{1}$ in row 2, column 2 of μ. We have $L^u_{<a} = 1,1,2,3,2,4$, so $\omega(L^u_{<a}) = (2,2,1,1)$. Thus $|a|' + \omega(L^u_{<a})_{|a|} = (d+1-(1))+(2,2,1,1)_1 = 4+2 = 6$. Also, $|a|'+c(a)-r(a) = (d+1-(1))+2-2 = 4$. Therefore the contribution of this entry to c_L is $y_6 - y_4$.*

Similarly, one computes the contribution of the entry $\overline{2}$ in row 1, column 3 to be $y_5 - y_5$ and the contribution of the entry $\overline{3}$ in row 2, column 1 to be $y_3 - y_1$. Therefore $c_L = (y_5 - y_5)(y_6 - y_4)(y_3 - y_1)$, which equals 0.

2.6. **Nonnegativity and Positivity.** If $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, then we write $c_L > 0$ if each factor in (3) is of the form $y_i - y_j$ with $i > j$. We write $c_L \geq 0$ if either $c_L > 0$ or $c_L = 0$. Note that the definition of $c_L \geq 0$ does not preclude the possibility that some factor of (3) is of the form $y_i - y_j$ with $i < j$; however, in this case some other factor must be of the form $y_i - y_j$ with $i = j$, thus forcing $c_L = 0$. The following proposition is proven in Section 4.

Proposition 2.7. *If $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, then $c_L \geq 0$.*

By Theorem 2.4, if $c_L = 0$ for all $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, then $c^{\nu}_{\lambda,\mu} = 0$. Proposition 2.7 implies that the converse is true as well:

Corollary 2.8. *If $c^{\nu}_{\lambda,\mu} = 0$, then $c_L = 0$ for all $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$.*

Proof. Denote by $c_L|_{y_i=i}$ the integer obtained by specializing each y_i to i in c_L. By Proposition 2.7, $c_L|_{y_i=i} \geq 0$, and $c_L|_{y_i=i} = 0$ if and only if $c_L = 0$. By Theorem 2.4, $\sum_{L\in\mathcal{LR}^{\nu}_{\lambda,\mu}} c_L = c^{\nu}_{\lambda,\mu} = 0$. Thus $\sum_{L\in\mathcal{LR}^{\nu}_{\lambda,\mu}} (c_L|_{y_i=i}) = 0$, which implies that $c_L|_{y_i=i} = 0$ for all $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, which in turn implies $c_L = 0$ for all $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$. $\square$

In Example 2.5, $\mu \not\subseteq \nu$. Thus by (2), $c^{\nu}_{\lambda,\mu} = 0$. Hence Corollary 2.8 verifies $c_L = 0$.

Let $\mathcal{LR}^{\nu\,+}_{\lambda,\mu}$ be the set of $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$ for which $c_L > 0$. By Proposition 2.7, we can restrict the summation in Theorem 2.4 to such L:

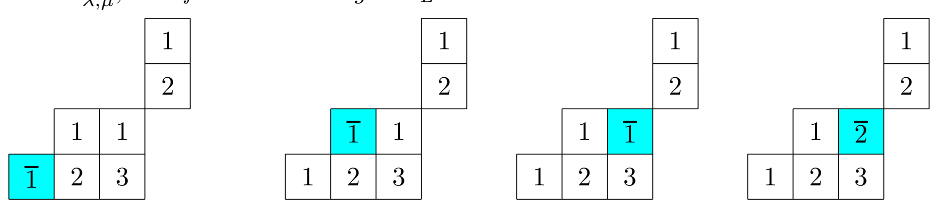
Corollary 2.9. $c^{\nu}_{\lambda,\mu} = \sum_{L \in \mathcal{LR}^{\nu\,+}_{\lambda,\mu}} c_L.$

One could, of course, use (3), the definition of c_L, to distinguish between $c_L > 0$ and $c_L = 0$: $c_L > 0$ if and only if $\omega(L^u_{<a})_{|a|} > c(a) - r(a)$ for all barred $a \in L$. The following Proposition gives a number of other tests for more efficiently making this determination.

Proposition 2.10. *If $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, then the following are equivalent:*
1. $c_L > 0$
2. $\omega(L^u_{<a})_{|a|} > c(a) - r(a)$ *for all barred* $a \in L$.
3. $\omega(L^u_{<a})_{|a|} > c(a) - r(a)$ *for all barred* $a \in L$ *with* $r(a) = 1$.
4. $\omega(L^u_{<a})_{|a|} \geq c(a)$ *for all barred* $a \in L$
5. $\omega(L^u_{<a})_{|a|} \geq c(a)$ *for all barred* $a \in L$ *with* $r(a) = 1$.

If $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$ satisfies any of these equivalent conditions, then we say that L is **positive**. It is obvious that $4 \implies 2 \implies 3 \iff 5$. In Section 4, we prove $3 \implies 4$. Condition 3 states that it suffices to check barred entries on the bottom row of L for positivity. Condition 4 has the following interpretation: for any barred entry $a \in L$, the corresponding factor $y_i - y_j$ in c_L satisfies $i - j \geq r(a)$ (which of course implies $i - j > 0$, the condition required for positivity).

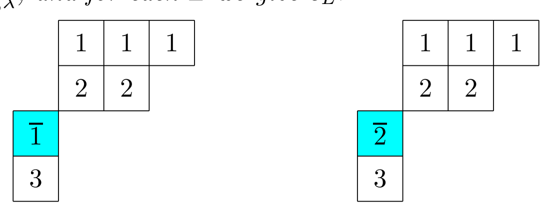
Example 2.11. *Let $d = 3$, $\lambda = (1,1)$, $\mu = (3,2)$, and $\nu = (3,2,1)$. We list all $L \in \mathcal{LR}^{\nu\,+}_{\lambda,\mu}$, and for each L we give c_L:*



$c_L = y_6 - y_5$ $c_L = y_5 - y_3$ $c_L = y_4 - y_2$ $c_L = y_3 - y_1$

Note that if L has an unbarred 2 in the upper right box of μ, then the unbarred column word of L is not Yamanouchi, and if L has two unbarred 1's on the top row of μ and is not the leftmost diagram, then $c_L = 0$; thus we do not include such L among $\mathcal{LR}^{\nu\,+}_{\lambda,\mu}$. By Corollary 2.9, $c^{\nu}_{\lambda,\mu} = (y_6 - y_5) + (y_5 - y_3) + (y_4 - y_2) + (y_3 - y_1)$.

We list all $L \in \mathcal{LR}^{\nu\,+}_{\mu,\lambda}$, and for each L we give c_L:



$c_L = y_6 - y_2$ $c_L = y_4 - y_1$

By Corollary 2.9, $c^{\nu}_{\mu,\lambda} = (y_6 - y_2) + (y_4 - y_1)$. We see that $c^{\nu}_{\mu,\lambda} = c^{\nu}_{\lambda,\mu}$. This is a general fact ensured by (1); however, it is not apparent from the statement of Corollary 2.9.

See also Example 10.3, where these same coefficients $c^{\nu}_{\lambda,\mu}$ are computed using the Molev-Sagan rule.

Example 2.12. *For cases where $\mu = \nu$, a formula for $c^{\nu}_{\lambda,\mu}$ which produces a different positive expression than Corollary 2.9 appears in* [Bi], [IN], *and* [Kr1]. *For example, using this formula, for $d = 3$, $\lambda = (2,1)$, and $\mu = \nu = (3,3,1)$, one computes:*

$$c^{\nu}_{\lambda,\mu} = (y_6 - y_1)(y_6 - y_3)(y_5 - y_1) + (y_6 - y_1)(y_5 - y_4)(y_5 - y_1).$$

Using Corollary 2.9:

$$\begin{aligned} c^{\nu}_{\lambda,\mu} =& (y_5 - y_3)(y_5 - y_1)(y_3 - y_1) + (y_6 - y_4)(y_5 - y_1)(y_3 - y_1) \\ &+ (y_6 - y_4)(y_6 - y_3)(y_3 - y_1) + (y_5 - y_3)(y_4 - y_3)(y_5 - y_1) \\ &+ (y_6 - y_4)(y_4 - y_3)(y_5 - y_1) + (y_6 - y_4)(y_6 - y_3)(y_4 - y_3) \\ &+ (y_6 - y_4)(y_5 - y_4)(y_5 - y_1) + (y_6 - y_4)(y_5 - y_4)(y_6 - y_3) \end{aligned}$$

$$\begin{aligned} c^{\nu}_{\mu,\lambda} =& (y_6 - y_4)(y_6 - y_2)(y_5 - y_2) + (y_5 - y_3)(y_6 - y_2)(y_5 - y_2) \\ &+ (y_6 - y_4)(y_6 - y_2)(y_2 - y_1) + (y_5 - y_3)(y_6 - y_2)(y_2 - y_1) \\ &+ (y_6 - y_4)(y_5 - y_1)(y_2 - y_1) + (y_5 - y_3)(y_5 - y_1)(y_2 - y_1). \end{aligned}$$

These three polynomials are, of course, equal.

For $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, $|\lambda| + |\mu| - |\nu| = \#(\text{entries of } L) - \#(\text{unbarred entries of } L) = \#(\text{barred entries of } L)$ which equals $\deg(c^{\nu}_{\lambda,\mu})$ if $c^{\nu}_{\lambda,\mu} \neq 0$. If $|\lambda| + |\mu| - |\nu| = 0$, then L has no barred entries, and Theorem 2.4 reduces to a version of the classical Littlewood-Richardson rule (see [F1], [LR], [Sa]).

2.13. **Defining the Structure Constants $C^{\nu}_{\lambda,\mu}$ for products of two Schubert Classes in $H^*_T(Gr_{d,n})$.** The **Grassmannian** $Gr_{d,n}$ is the set of d-dimensional complex subspaces of $\mathbb{C}^n$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$. Consider the opposite standard flag, whose i-th space is $\mathrm{Span}(e_n, \ldots, e_{n-i+1})$. For $\lambda \in \mathcal{P}_{d,n}$, the (opposite) Schubert variety X_λ of $Gr_{d,n}$ is defined by incident relations:

$$X_\lambda = \{V \in Gr_{d,n} \mid \dim(V \cap F_i) \geq \dim(\mathbb{C}^\lambda \cap F_i)\},\ i = 1, \ldots, n,$$

where $C^\lambda = \mathrm{Span}(e_{\lambda_d + d}, \ldots, e_{\lambda_1 + 1})$. The Schubert variety X_λ is invariant under the action of the group $T = (\mathbb{C}^*)^n$ on $Gr_{d,n}$. Thus it determines a class S_λ in the equivariant cohomology ring $H^*_T(Gr_{d,n})$.

Let $V = Gr_{d,n} \times \mathbb{C}^n$ be the trivial vector bundle on $Gr_{d,n}$, with diagonal T-action, where T acts naturally on $Gr_{d,n}$ and on $\mathbb{C}^n$ (thus V is not equivariantly trivial). Let $Y_1, \ldots, Y_n$ be the equivariant Chern roots of V^*. Then $Y_1, \ldots, Y_n \in H^*_T(Gr_{d,n})$ are algebraically independent, and $H^*_T(Gr_{d,n})$ is a free $\mathbb{Z}[Y_1, \ldots, Y_n]$-module, with the Schubert classes forming a $\mathbb{Z}[Y_1, \ldots, Y_n]$-basis. Thus for $\lambda, \mu \in \mathcal{P}_{d,n}$,

$$S_\lambda S_\mu = \sum_{\nu \in \mathcal{P}_{d,n}} C^{\nu}_{\lambda,\mu} S_\nu,\ \text{for some } C^{\nu}_{\lambda,\mu} \in \mathbb{Z}[Y_1, \ldots, Y_n].$$

Let $S = \{(w, v) \in V \mid v \in w\}$ be the tautological vector bundle on $Gr_{d,n}$, a T-invariant sub-bundle of V, and let $X_1, \ldots, X_d$ the equivariant Chern roots of S. We have (see [F2], [KT], [Mi])

Proposition 2.14. *For $\lambda \in \mathcal{P}_d$, $S_\lambda = s_\lambda(X_1, \ldots, X_d, -Y_n, \ldots, -Y_1, 0, 0, \ldots)$.*

Thus by specializing (1), we can determine the structure constants $C^{\nu}_{\lambda,\mu}$.

Corollary 2.15. *For $\lambda,\mu,\nu \in \mathcal{P}_{d,n}$, $C^{\nu}_{\lambda,\mu} = c^{\nu}_{\lambda,\mu}(-Y_n,\ldots,-Y_1,0,0,\ldots)$.*

2.16. **Computing the Structure Constants $C^{\nu}_{\lambda,\mu}$.** Let $\lambda,\mu,\nu \in \mathcal{P}_{d,n}$. By Corollary 2.15, $C^{\nu}_{\lambda,\mu}$ can be computed using the formula for $c^{\nu}_{\lambda,\mu}$. Letting $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, we have:

(4) Both subscripts in equation (3) for c_L lie between 1 and n.

Indeed,

(i) $|a|' + \omega(L^u_{<a})_{|a|} = d+1-|a| + \omega(L^u_{<a})_{|a|} \le d + \omega(L^u_{<a})_{|a|} \le d+\nu_1 \le n$. The last two inequalities are due to $\omega(L^u) = \nu$ and $\nu \in \mathcal{P}_{d,n}$ respecctively.

(ii) $|a|' + c(a) - r(a) = d+1-|a| + c(a) - r(a) < d + c(a) \le d + \mu_1 \le n$. The last two inequalities are due to the facts that the reverse Young diagram μ has μ_1 columns and $\mu \in \mathcal{P}_{d,n}$ respectively.

(iii) One checks that $|a| \le d+1-r(a)$. Thus $|a|' + c(a) - r(a) = d+1-|a| + c(a) - r(a) \ge c(a) \ge 1$.

Define

$$C_L = c_L(-Y_n,\ldots,-Y_1,0,0,\ldots) \tag{5}$$

$$= \prod_{\substack{a\in L\\ a \text{ barred}}} \left(Y_{(n-d)+|a|-(c(a)-r(a))} - Y_{(n-d)+|a|-\omega(L^u_{<a})_{|a|}} \right). \tag{6}$$

We write $C_L > 0$ if each factor in (6) is of the form $Y_i - Y_j$ with $i > j$, and we write $C_L \ge 0$ if either $C_L > 0$ or $C_L = 0$. By (4), (5), and the algebraic independence of the Y_i's, $c_L = 0 \iff C_L = 0$, and $c_L > 0 \iff C_L > 0$. Thus Propositions 2.7 and 2.10 imply

Corollary 2.17. *$C_L \ge 0$, and $C_L > 0 \iff L$ satisfies any of the equivalent conditions of Proposition 2.10.*

By Theorem 2.4, Corollary 2.15, and Corollary 2.17, we have

Corollary 2.18. $C^{\nu}_{\lambda,\mu} = \sum_{L \in \mathcal{LR}^{\nu}_{\lambda,\mu}} C_L = \sum_{L \in \mathcal{LR}^{\nu\,+}_{\lambda,\mu}} C_L.$

Example 2.19. *We continue Example 2.11. For $n \ge 6$, $\lambda,\mu \in \mathcal{P}_{d,n}$. Thus for $\nu \in \mathcal{P}_{d,n}$, $C^{\nu}_{\lambda,\mu} = (Y_{n+1-2} - Y_{n+1-6}) + (Y_{n+1-1} - Y_{n+1-4}) = (Y_{n-1} - Y_{n-5}) + (Y_n - Y_{n-3})$.*

2.20. **Equivalence of Molev's Results.** Our equivariant Littlewood-Richardson skew tableaux are in bijection with Molev's indexing tableaux [Mo1]. To determine the tableau in [Mo1] which corresponds to our $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, replace all barred entries of $L|_{\mu}$ by unbarred entries and vice versa, and then rotate the resulting object by 180 degrees. If one makes this modification, then Corollary 2.9 is equivalent to [Mo1, Theorem 2.1] after accounting for the relationship between double Schur functions and factorial Schur functions (see [Mo1, (1.9)]), and Corollary 2.18 is identical to [Mo1, Corollary 3.1].

In our notation, Molev's positivity criterion states that for $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, $c_L > 0$ if and only if

$$\omega(L^u)'_{c(a)} \ge |a| \text{ for all } a \in L \text{ with } r(a) = 1, \tag{7}$$

where $\omega(L^u)'$ is the *conjugate partition* to $\omega(L^u)$ (in this case Molev calls $L|_\mu$ ν-*bounded*). One can re-express (7) as follows:

$$\omega(L^u)_{|a|} \geq c(a) \text{ for all } a \in L \text{ with } r(a) = 1.$$

It is not difficult to see that this condition is equivalent to Proposition 2.10.5.

Related and more general results have been achieved in several directions. Robinson [R] has given a Pieri rule in the equivariant cohomology of the flag variety. McNamara [Mc] introduced factorial Grothendieck polynomials, generalizations of factorial Schur functions, and has given a rule for computing the structure constants for various of their products.

This paper is organized as follows. In Section 3, we introduce various types of tableaux which will appear throughout the paper. In Section 4, we prove Propositions 2.7 and 2.10, the nonnegativity property and positivity criteria of c_L. In Section 5, we outline the main steps in our proof of Theorem 2.4, whose two difficult technical lemmas are proved in Sections 7 and 8. In Section 6, we define a set of involutions required for the proofs of these two lemmas. In Section 9, we describe a weight preserving bijection between equivariant Littlewood-Richardson skew tableaux and trapezoid puzzles, which restricts to a bijection between positive equivariant Littlewood-Richardson skew tableaux and Knutson-Tao puzzles. In Section 10, we recall the Molev-Sagan rule.

3. Several Types of Tableaux

In this section we collect the definitions of the several types of tableaux which we will encounter in the remainder of the paper: reverse barred tableaux, reverse barred subtableaux, and reverse hatted tableaux. The latter two are refinements of the first.

A **reverse barred tableau of shape** μ is a skew barred tableau of shape $\emptyset * \mu$; alternatively, it can be defined as a reverse Young diagram of shape μ, each of whose boxes is filled with either an integer k or a barred integer $\overline{k}$, $k \in \{1, \dots, d\}$, in such a way that the values of the entries, without regard to whether or not they are barred, weakly increase along any row from left to right and strictly increase along any column from top to bottom. We denote the set of all reverse barred tableaux of shape μ by $\mathcal{B}(\mu)$. If $B \in \mathcal{B}(\mu)$, then define $\lambda * B$ to be the skew barred tableau obtained by placing the Young tableau whose i-th row consists of λ_i i's above and to the right of B. Then $B \mapsto \lambda * B$ defines a bijection from $\{B \in \mathcal{B}(\mu) \mid (\lambda * B)^u$ is Yamanouchi$\}$ to the equivariant Littlewood-Richardson skew tableaux of shape $\lambda * \mu$, whose inverse map is $L \mapsto L|_\mu$. Any $a \in B$ also corresponds to an entry $a \in \lambda * B$. Define B^u and $B^u_{<a}$ to be $(\emptyset * B)^u$ and $(\emptyset * B)^u_{<a}$ respectively.

A **reverse barred subtableaux of shape** μ is a reverse Young diagram μ each of whose boxes contains either an integer k, a barred integer $\overline{k}$, or is empty, where $k \in \{1, \dots, d\}$. A **reverse subtableau of shape** μ is a reverse barred tableau of shape μ which has no barred entries. We do not define any notion of row semistrictness or column strictness for such objects, as no such conditions will be required for our purposes. Denote the set of all reverse subtableaux and reverse barred subtableaux of shape μ by $\mathcal{R}_{sub}(\mu)$ and $\mathcal{B}_{sub}(\mu)$ respectively. We have the following containments:

$$\begin{array}{ccc} \mathcal{R}_{sub}(\mu) & \subset & \mathcal{B}_{sub}(\mu) \\ \cup & & \cup \\ \mathcal{R}(\mu) & \subset & \mathcal{B}(\mu) \end{array}$$

For $B \in \mathcal{B}_{sub}(\mu)$ and $a \in B$, define B^u and $B^u_{<a}$ just as for elements of $\mathcal{B}(\mu)$, assuming that when reading the unbarred column word of B, both barred entries and empty boxes are skipped over. If $B \in \mathcal{B}_{sub}(\mu)$, then define $\widetilde{B} \in \mathcal{R}_{sub}(\mu)$ to be the reverse subtableau obtained by removing all bars from entries of B, i.e., replacing each barred entry of B by an unbarred entry of the same value.

A **reverse hatted tableau of shape** μ is a reverse Young diagram μ each of whose boxes is filled with either a(n) (un-hatted) integer k, a *left* hatted integer $\check{k}$, or a *right* hatted integer $\hat{k}$, $k \in \{1, \ldots, d\}$, such that the values of the entries, without regard to whether or not they are hatted, weakly increase along any row from left to right and strictly increase along any column from top to bottom. Denote the set of all reverse hatted tableaux of shape μ by $\mathcal{H}(\mu)$. If H is a reverse hatted tableau, then define $\overline{H}$ to be the reverse barred tableau produced by replacing all hats (right and left) by bars. Hence for a reverse barred tableau B with m barred entries, there are 2^m reverse hatted tableaux H such that $\overline{H} = B$ (since each $\overline{k}$ of B can be replaced by either $\check{k}$ or $\hat{k}$). For $a \in H$, define H^u and $H^u_{<a}$ to be $\overline{H}^u$ and $\overline{H}^u_{<a}$ respectively. Define H^l (resp. H^r) to be the set of left-hatted (resp. right-hatted) entries of H.

We next give two different ways to generalize the polynomial c_L defined in Section 2. Let $\xi \in \mathbb{N}^d$. For $B \in \mathcal{B}_{sub}(\mu)$, define

$$c_{\xi,B} = \prod_{\substack{a \in B \\ a \text{ barred}}} (y_{e_{\xi,B}(a)} - y_{f_B(a)}), \tag{8}$$

where $e_{\xi,B}(a) := (\xi + \omega(B^u_{<a}))_{|a|}$ and $f_B(a) := |a|' + c(a) - r(a)$, $a \in B$. For $H \in \mathcal{H}(\mu)$, define

$$d_{\xi,H} = \prod_{a \in H^l} y_{e_{\xi,H}(a)} \prod_{a \in H^r} (-y_{f_H(a)}), \tag{9}$$

where $e_{\xi,H}(a) := (\xi + \omega(H^u_{<a}))_{|a|}$ and $f_H(a) := |a|' + c(a) - r(a)$, $a \in H$. In both (8) and (9), the empty product is defined to equal 1.

Let $B \in \mathcal{B}(\mu)$. By definition,

$$c_{\lambda * B} = c_{\lambda+\rho+\mathbf{1},B}. \tag{10}$$

In addition, the equation

$$c_{\xi,B} = \sum_{\substack{H \in \mathcal{H}(\mu) \\ \overline{H} = B}} d_{\xi,H} \tag{11}$$

expresses $c_{\xi,B}$ by expanding (8) in terms of monomials in the y_i's. Combining (10) and (11), we have

$$c_{\lambda * B} = \sum_{\substack{H \in \mathcal{H}(\mu) \\ \overline{H} = B}} d_{\lambda+\rho+\mathbf{1},H}. \tag{12}$$

If $R \in \mathcal{R}_{sub}(\mu)$, then define $(x \,|\, y)^R = \prod_{a \in R} \left(x_a - y_{f_R(a)}\right)$. This definition is consistent with the definition of $(x \,|\, y)^R$, $R \in \mathcal{R}(\mu)$, given in Section 2.

4. Proofs of Nonnegativity Property and Positivity Criteria

Let $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, and let $B = L|_{\mu}$. For $a \in B$, which we also view as an entry of L, define $L^u_{\leq a}$ to be $L^u_{<a}$ if a is barred, or $L^u_{<a}$ appended with a if a is not barred. Define

$$\Delta(a) = \omega(L^u_{\leq a})_{|a|} - c(a) + r(a).$$

If a is barred, then $\omega(L^u_{\leq a}) = \omega(L^u_{<a})$; hence $\Delta(a)$ gives the difference between the two indices $i - j$ of the factor $y_i - y_j$ corresponding to a in (3). Therefore Propositions 2.7 and 2.10 are equivalent to the following two lemmas respectively.

Lemma 4.1. *If $\Delta(a) < 0$ for some barred $a \in B$, then $\Delta(b) = 0$ for some barred $b \in B$.*

Lemma 4.2. *The following are equivalent:*
(i) $\Delta(a) > 0$ for all barred $a \in B$.
(ii) $\Delta(a) > 0$ for all barred $a \in B$ with $r(a) = 1$.
(iii) $\Delta(a) \geq r(a)$ for all barred $a \in B$.

Before proving these two lemmas, we first establish some properties of Δ.

Lemma 4.3. *The function $\Delta : B \to \mathbb{Z}$ satisfies the following properties:*
(i) If $a \in B$ and $c(a) = 1$, then $\Delta(a) \geq 0$, with equality implying that a is barred.
(ii) If one moves left by one box, then Δ can decrease by at most one. If it does decrease by one, then the left box must be barred.
(iii) If $\Delta(a) \leq 0$ for some $a \in B$, then $\Delta(b) = 0$ for some barred $b \in B$ on the same row as a.
(iv) The function $a \mapsto \Delta(a) - r(a)$ is weakly decreasing as one moves down along any column.

Proof. (i) Since $r(a) \geq 1$, $\Delta(a) \geq 0$. If $\Delta(a) = 0$, then $r(a) = 1$ and $\omega(L^u_{\leq a})_{|a|} = 0$. The latter requirement implies that a is barred.
(ii) If entry m lies one box left of a, then $-c(m) = -c(a) - 1$, $r(m) = r(a)$, and $\omega(L^u_{\leq m})_{|m|} \geq \omega(L^u_{\leq a})_{|m|} \geq \omega(L^u_{\leq a})_{|a|}$, where the first inequality is an equality if and only if m is barred. The second inequality is a consequence of the fact that the unbarred column word of L is Yamanouchi.
(iii) Let m be rightmost entry in the same row as a. If $\Delta(m) = 0$, then by (i), m is barred, so letting $b = m$ we are done. Otherwise $\Delta(m) > 0$. By (ii), as one moves left from m to a along the row the two entries lie on, one must encounter some barred b for which $\Delta(b) = 0$.
(iv) If entry m lies one box below a, then $\omega(L^u_{\leq a})_{|a|} = \omega(L^u_{\leq m})_{|a|} \geq \omega(L^u_{\leq m})_{|m|}$, since the unbarred column word of L is Yamanouchi. □

Proof of Lemmas 4.1 and 4.2. Lemma 4.1 is a special case of Lemma 4.3(iii). In Lemma 4.2, implications (iii) $\implies$ (i) $\implies$ (ii) are clear. We prove (ii) $\implies$ (iii). Suppose that $a \in B$ is a barred entry such that $\Delta(a) < r(a)$. Let m be the bottom entry in column $c(a)$. By Lemma 4.3(iv), $\Delta(m) < r(m)$. Since $r(m) = 1$, $\Delta(m) \leq 0$. By Lemma 4.3(iii), $\Delta(b) = 0$ for some barred b on the bottom row of B. □

5. Generalization of Stembridge's Proof

In this section we list the main steps in the proof of Theorem 2.4. The bulk of the technical work, however, namely the proofs of Lemmas 5.1 and 5.2, is taken up in the three subsequent sections. The underlying logic and structure of our arguments in this and the following three sections follows Stembridge [St], who works out similar results for ordinary Schur functions.

For $k \in \mathbb{N}$, define the polynomial $(x_j \,|\, y)^k = (x_j - y_1) \cdots (x_j - y_k)$. For $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{N}^d$, define $a_\xi(x \,|\, y) = \det[(x_j \,|\, y)^{\xi_i}]_{1 \le i,j \le d}$.

Lemma 5.1. $a_{\lambda+\rho}(x \,|\, y) s_\mu(x \,|\, y) = \displaystyle\sum_{B \in \mathcal{B}(\mu)} c_{\lambda * B} a_{\lambda+\rho+\omega(B^u)}(x \,|\, y).$

Lemma 5.2. $\sum c_{\lambda * B} a_{\lambda+\rho+\omega(B^u)}(x \,|\, y) = 0$, *where the sum is over all $B \in \mathcal{B}(\mu)$ such that the unbarred column word of $\lambda * B$ is not Yamanouchi.*

The following three corollaries follow easily from these two lemmas.

Corollary 5.3. $a_{\lambda+\rho}(x \,|\, y) s_\mu(x \,|\, y) = \sum c_{\lambda * B} a_{\lambda+\rho+\omega(B^u)}(x \,|\, y)$, *where the sum is over all $B \in \mathcal{B}(\mu)$ such that the unbarred column word of $\lambda * B$ is Yamanouchi.*

Suppose that $B \in \mathcal{B}(\mu)$ is such that the unbarred column word of $\emptyset * B$ is Yamanouchi. If B has barred entries, then by Propositions 2.7 and 2.10.5, $c_{\emptyset * B} = 0$. If B has no barred entries, then B must be the unique reverse tableau of shape μ and content μ: B contains a 1 at the top of each column, and its entries increase by 1 per box as one moves down any column. Thus, by setting $\lambda = \emptyset$ in Corollary 5.3, we arrive at a new proof of the bialternant formula for the factorial Schur function ([GG], [Ma1]):

Corollary 5.4. $s_\mu(x \,|\, y) = a_{\mu+\rho}(x \,|\, y)/a_\rho(x \,|\, y)$.

Dividing both sides of the equation in Corollary 5.3 by $a_\rho(x \,|\, y)$ and applying Corollary 5.4 yields

Corollary 5.5. $s_\lambda(x \,|\, y) s_\mu(x \,|\, y) = \sum c_{\lambda * B} s_{\lambda+\omega(B^u)}(x \,|\, y)$, *where the sum is over all $B \in \mathcal{B}(\mu)$ such that the unbarred column word of $\lambda * B$ is Yamanouchi.*

Regrouping the terms in this summation:

$$s_\lambda(x \,|\, y) s_\mu(x \,|\, y) = \sum_\nu \left(\sum_{\substack{B \in \mathcal{B}(\mu) \\ (\lambda * B)^u \text{ Yamanouchi} \\ \lambda + \omega(B^u) = \nu}} c_{\lambda * B} \right) s_\nu(x \,|\, y) = \sum_\nu \left(\sum_{L \in \mathcal{LR}^\nu_{\lambda,\mu}} c_L \right) s_\nu(x \,|\, y).$$

This proves Theorem 2.4.

Remark 5.6. *Let $\kappa \in \mathcal{P}_d$, $\kappa \le \mu$, i.e., $\kappa_i \le \mu_i$, $i = 1, \ldots, d$. One can extend our analysis to factorial skew Schur functions of the form $s_{\mu/\kappa}(x \,|\, y)$ (see* [Ma1]*). One replaces $\mathcal{B}(\mu)$ with $\mathcal{B}(\mu/\kappa)$, the set of all reverse barred tableaux of shape μ/κ. All above definitions extend naturally. For example, for $B \in \mathcal{B}(\mu/\kappa)$, $c_{\lambda * B}$ is computed just as for $B \in \mathcal{B}(\mu)$, but with all boxes of $\kappa \subset \mu$ assumed to be empty. All proofs are virtually unchanged, modified only by formally replacing μ by μ/κ. As a generalization of Corollary 5.5, we obtain*

$$s_\lambda(x \,|\, y) s_{\mu/\kappa}(x \,|\, y) = \sum c_{\lambda * B} s_{\lambda+\omega(B^u)}(x \,|\, y),$$

where the sum is over all $B \in \mathcal{B}(\mu/\kappa)$ such that $(\lambda * B)^u$ is Yamnaouchi. This generalizes Zelevinsky's extension of the Littlewood-Richardson rule ([St], [Z]*).*

6. Involutions on Reverse Hatted Tableaux

In his proof, Stembridge [St] utilizes involutions on Young tableaux introduced by Bender and Knuth [BK]. There is an analogous set of involutions on $\mathcal{H}(\mu)$ which satisfy properties required for the proofs of Lemmas 5.1 and 5.2 (see Lemma 6.4). We remark that we were unable to find a suitable set of involutions on $\mathcal{B}(\mu)$, and this is what initially led us to examine $\mathcal{H}(\mu)$. If the involutions on $\mathcal{H}(\mu)$ are restricted to $\mathcal{R}(\mu)$, then the Bender-Knuth involutions are recovered.

6.1. **The Involutions $\mathbf{s_1}, \ldots, \mathbf{s_{d-1}}$ of $\mathcal{H}(\mu)$.** Let $H \in \mathcal{H}(\mu)$, and let $i \in \{1, \ldots, d-1\}$ be fixed. Then an entry a of H with value i or $i+1$ is

- **free** if there is no entry of value $i+1$ or i respectively in the same column;
- **semi-free** if there is an entry of value $i+1$ or i respectively in the same column, and at least one of the two is hatted; or
- **locked** if there is an entry of value $i+1$ or i respectively in the same column, and both entries are unhatted.

Note that any entry of value i or $i+1$ must be exactly one of these three types, and each hatted entry of value i or $i+1$ must be either free or semi-free. In any row, the free entries are consecutive. Semi-free entries come in pairs, one below the other, as do locked entries.

To define the action of s_i on $H \in \mathcal{H}(\mu)$, we first consider how it modifies the free entries of H (see Example 6.2):

1. Let S be a maximal string of free entries with values i and $i+1$ on some row of H. Let S°, S^l, and S^r denote the unhatted, left-hatted, and right-hatted entries of S respectively. Modify $S^\circ \cup S^l$, as follows:
   A Change the value of each entry of value i to $i+1$ and each entry of value $i+1$ to i, without changing whether or not it has a left hat.
   B Swap the entries of value i with those of value $i+1$, as follows: remove all entries of value i; then move each entry of value $i+1$, beginning with the rightmost one, into the rightmost available empty box; then put the removed entries of value i back into the empty boxes of B, preserving the relative order of barred and unbarred entries.

   In this step, $S^\circ \cup S^l$ has been modified. No other entries of H, in particular no entries of S^r, have been modified, changed, or moved. Denote the modified string S by S_1. A potential problem has been introduced: the values of the entries of S_1 may not be weakly increasing as one moves from left to right. In step 2 we correct for this.
2. Let $(S_1^r)_i$ and $(S_1^r)_{i+1}$ denote the entries of S_1^r of value i and $i+1$ respectively. Beginning with the leftmost entry $a \in (S_1^r)_i$, let b be the entry of S_1 to the left of a. If b has value $i+1$, then switch the entries b and a, and then change the left entry from $\widehat{i}$ to $\widehat{i+1}$. Now move right to the next entry of $(S_1^r)_i$, and repeat this procedure until it has been performed on all entries of $(S_1^r)_i$. Next, beginning with the rightmost entry $a \in (S_1^r)_{i+1}$, let b be the entry of S_1 to the right of a. If b has value i, then switch the entries b and a, and then change the right entry from $\widehat{i+1}$ to $\widehat{i}$. Now move left to the next entry of $(S_1^r)_{i+1}$, and repeat this procedure until it has been performed on all entries of $(S_1^r)_{i+1}$.

   Upon completion, we denote by S_2 the resulting string obtained by modifying S_1. It is weakly increasing.

We next consider how s_i modifies the semi-free entries of H:

3. For a semi-free pair consisting of two entries lying in the same column of H, each entry removes its hat (if it has one) and places it on top of the other entry.

The reverse tableau s_iH is obtained by applying steps 1 and 2 to each maximal string S of free entries of H (replacing S by S_2) and then applying step 3 to each semi-free pair.

Example 6.2. *We illustrate steps 1 and 2. Suppose that $i = 2$, and S consists of the following maximal string of consecutive free entries lying along some row of H:*

$$\begin{array}{rcccccccccccc}
S = & 2 & \hat{2} & \check{2} & 2 & 2 & \check{2} & \hat{2} & 2 & 3 & \check{3} & \hat{3} & \check{3} \\
S^\circ \cup S^l = & 2 & & \check{2} & 2 & 2 & \check{2} & & 2 & 3 & \check{3} & & \check{3} \\
 & 3 & & \check{3} & 3 & 3 & \check{3} & & 3 & 2 & \check{2} & & \check{2} \\
 & 2 & & \check{2} & \check{2} & 3 & \check{3} & & 3 & 3 & \check{3} & & 3 \\
S_1 = & 2 & \hat{2} & \check{2} & \check{2} & 3 & \check{3} & \hat{2} & 3 & 3 & \check{3} & \hat{3} & 3 \\
S_2 = & 2 & \hat{2} & \check{2} & \check{2} & 3 & \hat{3} & \check{3} & 3 & 3 & \check{3} & \hat{3} & 3
\end{array}$$

In line 2 we remove the entries of S^r from the picture for convenience, in order to focus attention on the operations performed in step 1, which only affect $S^\circ \cup S^l$. In lines 3 and 4 the results of applying steps 1A and 1B successively to $S^\circ \cup S^l$ are shown. In line 5, the removed entries from S^r are replaced. In line 6, the result of applying step 2 to S_1 is shown. Only two entries are changed in this step.

This algorithm defines maps $b_l : H^l \to (s_iH)^l$ and $b_r : H^r \to (s_iH)^r$, as follows. If $a \in H$ and the value of a is neither i nor $i + 1$, then a remains unchanged in s_iH. Thus in this case, if $a \in H^l$ or $a \in H^r$, then we define $b_l(a) = a$ or $b_r(a) = a$ respectively. Assume the value of a is i or $i + 1$. If $a \in H^l$ is free, then in step 1A, the value of a is either increased or decreased by 1; in step 1B, it is then moved to a different box; in step 2, this new entry in this new box is moved at most one box and changed by at most one in value, resulting in the entry we denote by $b_l(a)$. If $a \in H^r$ is free, then a is unchanged in step 1 and moved at most one box and changed by at most one in value in step 2. Denote the resulting entry by $b_r(a)$. If $a \in H^l$ or $a \in H^r$ is semi-free, then $b_l(a)$ or $b_r(a)$ is the entry in s_iH which it gives its hat to.

In Example 6.2, if a is the rightmost entry of S, which is a $\check{3}$, then $b_l(a)$ is the $\check{2}$ which is the fourth entry of S_2 from left. These two entries are, of course, entries of H and s_iH respectively.

Lemma 6.3. *s_i is an involution on $\mathcal{H}(\mu)$, $i \in \{1, \dots, d-1\}$.*

Proof. We begin by showing that $s_iH \in \mathcal{H}(\mu)$, i.e., s_iH is row semistrict and column strict. The only nonobvious condition is that if S is any maximal string of free entries of H lying along some row, and S_2 the string that replaces it in s_iH, then s_iH weakly increases along the left and right boundaries of S_2. To see this, note that if any entry of H of value $i+1$ is free, then so are all entries of value $i+1$ to the right of it in the same row; and if any entry of H of value i is free, then so are all entries of value i to the left of it in the same row. Thus by the maximality of S, there are no entries of H of value i in the same row and to the right of S, and there are no entries of H of value $i+1$ in the same row and to the left of S. Hence changing values of S from i to $i+1$ and vice versa to form S_2 does not affect the row semistrictness of H along its boundaries.

We next show that $s_i^2 = \mathrm{id}$. Since the free entries of H lie in the same boxes as the free entries of s_iH, it suffices to show that $s_i^2(S) = S$ for any maximal string S of free entries of H (where s_iS is defined to be s_iH restricted to S). If step 1 is applied to $(s_iS)^\circ \cup (s_iS)^l$, then one sees that the same entries of $S^\circ \cup S^l$ are retrieved, although possibly not in their same boxes. However the relative order of the entries is the same. Now one checks that for $a \in H^r$, $b_r^2(a) = a$. □

Let σ_i be the simple transposition of the permutation group S_d which exchanges i and $i+1$. The involution s_i satisfies the following properties:

Lemma 6.4. *Let $H \in \mathcal{H}(\mu)$, $a \in H^l$, and $b \in H^r$. Then*
(i) $|b_l(a)| = \sigma_i|a|$
(ii) $\omega((s_iH)^u) = \sigma_i\omega(H^u)$.
(iii) $\omega((s_iH)^u_{<b_l(a)})_{|b_l(a)|} = \omega(H^u_{<a})_{|a|}$
(iv) $e_{\sigma_i\xi,s_iH}(b_l(a)) = e_{\xi,H}(a)$
(v) $f_{s_iH}(b_r(b)) = f_H(b)$
(vi) $d_{\sigma_i\xi,s_iH} = d_{\xi,H}$

Proof. If the value of a is not i or $i+1$, then (i), (iii), and (iv) are obvious. If the value of b is not i or $i+1$, then (v) is obvious. Thus we assume for these parts that the values of a and b are either i or $i+1$. Parts (i), (ii), and (iii) follow from the construction of s_i.
(iv) By parts (i) and (iii),

$$\begin{aligned} e_{\sigma_i\xi,s_iH}(b_l(a)) = (\sigma_i\xi + \omega((s_iH)^u_{<b_l(a)}))_{|b_l(a)|} &= (\sigma_i\xi)_{\sigma_i|a|} + \omega((s_iH)^u_{<a})_{|b_l(a)|} \\ &= (\xi + \omega(H^u_{<a}))_{|a|} = e_{\xi,H}(a). \end{aligned}$$

(v) Under b_r, the entry b is either kept in place, moved up, down, left, or right by one box. In these cases, its value is either left unchanged, decreased, increased, increased, or decreased by one respectively. The result now follows from the definition of f_H.
(vi) This is a consequence of (iv), (v), and (9). □

Let $H \in \mathcal{H}(\mu)$ and let $\sigma \in S_d$. Choose some decomposition of σ into simple transpositions: $\sigma = \sigma_{i_1} \cdots \sigma_{i_t}$. Define $\sigma H := s_{i_1} \cdots s_{i_t} H$. Although σH depends on the decomposition chosen for σ, by Lemma 6.4(ii) and (vi),

$$\omega((\sigma H)^u) = \sigma\omega(H^u) \qquad \text{and} \qquad d_{\sigma\xi,\sigma H} = d_{\xi,H}. \tag{13}$$

In particular, both $\omega((\sigma H)^u)$ and $d_{\sigma\xi,\sigma H}$ are independent of the decomposition of σ.

7. Proof of Lemma 5.1

Lemma 5.1 is a generalization of [St, (1)]. In proving [St, (1)], Stembridge uses the simple fact that if S is a tableau and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{N}^d$, then $x^\xi x^S = x^{\xi+\omega(S)}$. The generalization of this fact which we will need in order to prove Lemma 5.1 is the following lemma. Define $(x\,|\,y)^\xi = (x_1\,|\,y)^{\xi_1} \cdots (x_d\,|\,y)^{\xi_d}$.

Lemma 7.1. *Let $R \in \mathcal{R}_{sub}(\mu)$ and let $\xi \in \mathbb{N}^d$. Then*

$$(x \mid y)^{\xi}(x \mid y)^R = \sum_{\substack{B \in \mathcal{B}_{sub}(\mu) \\ \widetilde{B}=R}} c_{\xi+1,B} \cdot (x \mid y)^{\xi+\omega(B^u)}.$$

In fact, we only need this lemma for $R \in \mathcal{R}(\mu)$. We prove this result more generally for $R \in \mathcal{R}_{sub}(\mu)$ only to allow for induction on the number of entries of R (and thus allow for the possibility that some boxes of R are empty). We remark that $\mathcal{R}_{\text{sub}}(\mu)$ and $\mathcal{B}_{\text{sub}}(\mu)$ were introduced in this paper solely to allow for induction in this proof.

Proof. The proof is by induction on the number of entries in R. Let a be an entry of R with value k, such that R has no entry of value k in any column to the left of a. Let α be the box containing a. Let $R' = R \setminus a$ be the reverse subtableau which results from removing a from R.

If $B \in \mathcal{B}_{sub}(\mu)$ is such that $\widetilde{B} = R$, then the entry of B in box α, which we denote by B_α, must either be k or $\overline{k}$. Let B' denote $B \setminus B_\alpha$. The following three sets are in bijection with one another:

$$\begin{aligned}\{B \in \mathcal{B}_{sub}(\mu) \mid \widetilde{B} = R, B_\alpha = k\} &\longleftrightarrow \{B \in \mathcal{B}_{sub}(\mu) \mid \widetilde{B} = R, B_\alpha = \overline{k}\} \\ &\longleftrightarrow \{D \in \mathcal{B}_{sub}(\mu) \mid \widetilde{D} = R'\}.\end{aligned}$$

The first bijection simply adds a bar to B_α, and the second bijection removes B_α from B, mapping B to B'. For brevity, we denote $e_{B,\xi}(B_\alpha)$ and $f_B(B_\alpha)$ by just $e(B_\alpha)$ and $f(B_\alpha)$ respectively for the remainder of this proof. If B_α is unbarred, then

$$c_{\xi+1,B} = c_{\xi+1,B'} \quad \text{and} \quad (x \mid y)^{\xi+\omega(B^u)} = (x \mid y)^{\xi+\omega((B')^u)}(x_d - y_{e(B_\alpha)+1}).$$

On the other hand, if B_α is barred, then

$$c_{\xi+1,B} = c_{\xi+1,B'}(y_{e(B_\alpha)+1} - y_{f(B_\alpha)}) \quad \text{and} \quad (x \mid y)^{\xi+\omega(B^u)} = (x \mid y)^{\xi+\omega((B')^u)}.$$

Thus,

$$\begin{aligned}&\sum_{\substack{B \in \mathcal{B}_{sub}(\mu) \\ \widetilde{B}=R}} c_{\xi+1,B}(x \mid y)^{\xi+\omega(B^u)} \\ &\qquad = \sum_{\substack{B \in \mathcal{B}_{sub}(\mu) \\ \widetilde{B}=R \\ B_\alpha = k}} c_{\xi+1,B}(x \mid y)^{\xi+\omega(B^u)} + \sum_{\substack{B \in \mathcal{B}_{sub}(\mu) \\ \widetilde{B}=R \\ B_\alpha = \overline{k}}} c_{\xi+1,B}(x \mid y)^{\xi+\omega(B^u)} \\ &\qquad = \sum_{\substack{B \in \mathcal{B}_{sub}(\mu) \\ \widetilde{B}=R \\ B_\alpha = k}} c_{\xi+1,B'}(x \mid y)^{\xi+\omega((B')^u)}(x_{B_\alpha} - y_{e(B_\alpha)+1}) \\ &\qquad\qquad + \sum_{\substack{B \in \mathcal{B}_{sub}(\mu) \\ \widetilde{B}=R \\ B_\alpha = \overline{k}}} c_{\xi+1,B'}(y_{e(B_\alpha)+1} - y_{f(B_\alpha)})(x \mid y)^{\xi+\omega((B')^u)} \\ &\qquad = \sum_{\substack{B \in \mathcal{B}_{sub}(\mu) \\ \widetilde{B}=R}} \Big(c_{\xi+1,B'}(x \mid y)^{\xi+\omega((B')^u)}(x_{B_\alpha} - y_{e(B_\alpha)+1}) \\ &\qquad\qquad + \quad c_{\xi+1,B'}(y_{e(B_\alpha)+1} - y_{f(B_\alpha)})(x \mid y)^{\xi+\omega((B')^u)} \Big)\end{aligned}$$

$$= \sum_{\substack{B\in\mathcal{B}_{sub}(\mu)\\ \widetilde{B}=R}} c_{\xi+1,B'}(x\mid y)^{\xi+\omega((B')^u)}(x_{B_\alpha}-y_{f(B_\alpha)})$$
$$= \sum_{\substack{D\in\mathcal{B}_{sub}(\mu)\\ \widetilde{D}=R'}} \left(c_{\xi+1,D}(x\mid y)^{\xi+\omega(D^u)}\right)(x_{B_\alpha}-y_{f(B_\alpha)})$$
$$= (x\mid y)^{\xi}(x\mid y)^{R'}(x_{B_\alpha}-y_{f(B_\alpha)})$$
$$= (x\mid y)^{\xi}(x\mid y)^{R}.$$

□

Proof of Lemma 5.1.

$$a_{\lambda+\rho}(x\,|\,y)s_\mu(x\,|\,y) \overset{(a)}{=} \sum_{\sigma\in S_d}\sum_{R\in\mathcal{R}(\mu)} \operatorname{sgn}(\sigma)(x\mid y)^{\sigma(\lambda+\rho)}(x\mid y)^R$$
$$\overset{(b)}{=} \sum_{\sigma\in S_d}\sum_{R\in\mathcal{R}(\mu)}\sum_{\substack{B\in\mathcal{B}(\mu)\\ \widetilde{B}=R}} c_{\sigma(\lambda+\rho+\mathbf{1}),B}\operatorname{sgn}(\sigma)(x\mid y)^{\sigma(\lambda+\rho)+\omega(B^u)}$$
$$\overset{(c)}{=} \sum_{\sigma\in S_d}\sum_{R\in\mathcal{R}(\mu)}\sum_{\substack{B\in\mathcal{B}(\mu)\\ \widetilde{B}=R}}\sum_{\substack{H\in\mathcal{H}(\mu)\\ \overline{H}=B}} d_{\sigma(\lambda+\rho+\mathbf{1}),H}\operatorname{sgn}(\sigma)(x\mid y)^{\sigma(\lambda+\rho)+\omega(H^u)}$$
$$= \sum_{\sigma\in S_d}\sum_{H\in\mathcal{H}(\mu)} d_{\sigma(\lambda+\rho+\mathbf{1}),H}\operatorname{sgn}(\sigma)(x\mid y)^{\sigma(\lambda+\rho)+\omega(H^u)}$$
$$\overset{(d)}{=} \sum_{\sigma\in S_d}\sum_{H\in\mathcal{H}(\mu)} d_{\sigma(\lambda+\rho+\mathbf{1}),\sigma H}\operatorname{sgn}(\sigma)(x\mid y)^{\sigma(\lambda+\rho)+\omega((\sigma H)^u)}$$
$$\overset{(e)}{=} \sum_{\sigma\in S_d}\sum_{H\in\mathcal{H}(\mu)} d_{\lambda+\rho+\mathbf{1},H}\operatorname{sgn}(\sigma)(x\mid y)^{\sigma(\lambda+\rho+\omega(H^u))}$$
$$= \sum_{\sigma\in S_d}\sum_{B\in\mathcal{B}(\mu)}\sum_{\substack{H\in\mathcal{H}(\mu)\\ \overline{H}=B}} d_{\lambda+\rho+\mathbf{1},H}\operatorname{sgn}(\sigma)(x\mid y)^{\sigma(\lambda+\rho+\omega(H^u))}$$
$$\overset{(c)}{=} \sum_{\sigma\in S_d}\sum_{B\in\mathcal{B}(\mu)} c_{\lambda+\rho+\mathbf{1},B}\operatorname{sgn}(\sigma)(x\mid y)^{\sigma(\lambda+\rho+\omega(B^u))}$$
$$\overset{(a)}{=} \sum_{B\in\mathcal{B}(\mu)} c_{\lambda+\rho+\mathbf{1},B}\,a_{\lambda+\rho+\omega(B^u)}(x\,|\,y)$$
$$\overset{(f)}{=} \sum_{B\in\mathcal{B}(\mu)} c_{\lambda,B}\,a_{\lambda+\rho+\omega(B^u)}(x\,|\,y).$$

Equality (a) follows from the definition of a_μ, noting that $\sigma(\lambda+\rho)+1=\sigma(\lambda+\rho+\mathbf{1})$; (b) follows from Lemma 7.1, setting $S=R$ and $\xi=\sigma(\lambda+\rho)$; (c) from (11), with $\xi=\sigma(\lambda+\rho)$; (e) from (13); and (f) from (10). For (d), we use the fact that for a fixed σ and arbitrary decomposition $\sigma=\sigma_{i_1}\cdots\sigma_{i_t}$, since each s_{i_j} is an involution on $\mathcal{H}(\mu)$, as H runs over all elements of $\mathcal{H}(\mu)$, so does σH. □

8. Proof of Lemma 5.2

By (12), Lemma 5.2 is equivalent to the following lemma, whose statement and proof generalize arguments in [St]. For $H\in\mathcal{H}(\mu)$ and j a nonnegative integer, define $H_{<j}$ to be the sub-hatted tableau of H consisting of the portion of H lying in columns to the right of j, and $H^u_{<j} = (H_{<j})^u$ (and similarly for $H_{\leq j}$, $H_{>j}$, etc.).

Lemma 8.1. *Let $\lambda \in \mathcal{P}_n$. Then*

$$\sum d_{\lambda+\rho+\mathbf{1},H}\, a_{\lambda+\rho+\omega(H^u)}(x\,|\,y) = 0, \tag{14}$$

the sum being over all $H \in \mathcal{H}(\mu)$ for which $\lambda + \omega(H^u_{\leq j}) \notin \mathcal{P}_d$ for some j.

Proof. We call $H \in \mathcal{H}(\mu)$ for which $\lambda + \omega(H^u_{\leq j}) \notin \mathcal{P}_d$ for some j a *Bad Guy.* Let H be a Bad Guy, and let j be minimal such that $\lambda + \omega(H^u_{\leq j}) \notin \mathcal{P}_d$. Having selected j, let i be minimal such that $(\lambda + \omega(H^u_{\leq j}))_i < (\lambda + \omega(H^u_{\leq j}))_{i+1}$. Since $(\lambda + \omega(H^u_{\leq j-1}))_i \geq (\lambda + \omega(H^u_{\leq j-1}))_{i+1}$ (by the minimality of j), we must have $(\lambda + \omega(H^u_{\leq j-1}))_i = (\lambda + \omega(H^u_{\leq j-1}))_{i+1}$, and column j of H must have an unhatted $i+1$ but not an unhatted i. Thus

$$(\lambda + \rho + \mathbf{1} + \omega(H^u_{\leq j}))_i = (\lambda + \rho + \mathbf{1} + \omega(H^u_{\leq j}))_{i+1}. \tag{15}$$

Define H^* to be the reverse tableau of shape μ obtained from H by replacing $H_{>j}$ by $s_i(H_{>j})$, and leaving $H_{\leq j}$ unchanged. Notice first that H^* is still semistandard. Indeed, since σ_i applied to $H_{>j}$ can only change the values of its entries from i to $i{+}1$ and vice versa, the only possible violation of semistandardness of H^* would occur under the following scenario: (a) H has an entry a of value i in column j (which has to be either an $\hat{i}$ or $\check{i}$, and must lie directly above the entry $i{+}1$), (b) H has an entry b of value i immediately to the left of a, and (c) s_i applied to $H_{>j}$ changes the value of the entry in the position of entry b to $i+1$. However, this scenario is impossible. If (a) and (b) both hold, then since H is semistandard, the entry of H immediately below b must have value $i{+}1$ (we remark that the following property of the shape of a reverse tableau is critical here: if a reverse tableau contains three of the four boxes making up a square, namely the top-left, top-right, and bottom-right boxes, then it must contain the bottom-left box of the square as well). Therefore the entry in box b is not a free entry of $H_{>j}$, so s_i does not change its value, i.e., (c) is violated. Notice second that since $H^*_{\leq j} = H_{\leq j}$, we have that H^* is still a Bad Guy, and furthermore the map $H^* \mapsto H^{**}$ replaces $(H^*)_{>j}$ by $s_i((H^*)_{>j})$ and leaves $(H^*)_{\leq j}$ unchanged. Therefore $(H^{**})_{>j} = s_i((H^*)_{>j}) = s_i(s_i(H_{>j})) = H_{>j}$ (since s_i is an involution on reverse hatted tableaux; see Lemma 6.3), and $(H^{**})_{\leq j} = (H^*)_{\leq j} = H_{\leq j}$. Thus $H \mapsto H^*$ gives an involution on the set of Bad Guys of $H(\mu)$.

We define maps $b^*_l : H^l \to (H^*)^l$ and $b^*_r : H^r \to (H^*)^r$, as follows. If $a \in (H_{\leq j})^l$, then define $b^*_l(a) = a$. If $a \in (H_{>j})^l$, then during the construction of H^*, in the process of applying s_i to $H_{>j}$, a is mapped to $b_l(a) \in (H_{>j})^l$. This same element $b_l(a)$, regarded as an element of $(H^*)^l$, is denoted by $b^*_l(a)$. The map b^*_r is defined analogously.

We wish to show that for $a \in H^l$,

$$e_{\lambda+\rho+\mathbf{1},H}(a) = e_{\lambda+\rho+\mathbf{1},H^*}(b^*_l(a)), \tag{16}$$

and for $a \in H^r$,

$$f_H(a) = f_{H^*}(b^*_r(a)). \tag{17}$$

For $a \in (H_{\leq j})^l$ or $a \in (H_{\leq j})^r$, both (16) and (17) are obvious. The proof of (17) for $a \in (H_{>j})^r$ follows in much the same manner as the proof of Lemma 6.4(v).

It remains to prove (16) for $a \in (H_{>j})^l$. For such a, by Lemma 6.4(iii),

$$\omega({}_{j<}H^u_{<c(a)})_{|a|} = \omega({}_{j<}(H^*)^u_{<c(b^*_l(a))})_{|b^*_l(a)|}, \tag{18}$$

where ${}_{l<}H_{<m} := H_{>l} \cap H_{<m}$. By (15),

(19) $(\lambda+\rho+\mathbf{1}+\omega(H^u_{\leq j}))_{|a|} = (\lambda+\rho+\mathbf{1}+\omega(H^u_{\leq j}))_{\sigma_i|a|} = (\lambda+\rho+\mathbf{1}+\omega((H^*)^u_{\leq j}))_{\sigma_i|a|}$.

Thus

$$\begin{aligned} e_{\lambda+\rho+\mathbf{1},H}(a) &= \Big(\lambda+\rho+\mathbf{1}+\omega(H^u_{<c(a)})\Big)_{|a|} \\ &= \Big(\lambda+\rho+\mathbf{1}+\omega(H^u_{\leq j})+\omega({}_{j<}H^u_{<c(a)})\Big)_{|a|} \\ &\overset{(a)}{=} \big(\lambda+\rho+\mathbf{1}+\omega((H^*)^u_{\leq j})\big)_{\sigma_i|a|} + \omega\big({}_{j<}(H^*)^u_{<c(b^*_l(a))}\big)_{|b^*_l(a)|} \\ &\overset{(b)}{=} \Big(\lambda+\rho+\mathbf{1}+\omega((H^*)^u_{<c(b^*_l(a))})\Big)_{|b^*_l(a)|} \\ &= e_{\lambda+\rho+\mathbf{1},H^*}(b^*_l(a)). \end{aligned}$$

Equality (a) follows from (18) and (19); (b) follows from Lemma 6.4(i). This completes the proofs of (16) and (17).

Now (16) and (17) imply

$$\begin{aligned} d_{\lambda+\rho+\mathbf{1},H} &= \prod_{a\in H^l} y_{e_{\lambda+\rho+\mathbf{1},H}(a)} \prod_{a\in H^r}(-y_{f_H(a)}) \\ &= \prod_{a\in H^l} y_{e_{\lambda+\rho+\mathbf{1},H^*}(b^*_l(a))} \prod_{a\in H^r}(-y_{f_{H^*}(b^*_l(a))}) \\ &= \prod_{a\in (H^*)^l} y_{e_{\lambda+\rho+\mathbf{1},H^*}(a)} \prod_{a\in (H^*)^r}(-y_{f_{H^*}(a)}) = d_{\lambda+\rho+\mathbf{1},H^*}. \end{aligned} \tag{20}$$

By $\sigma_i\omega(H^u_{>j}) = \omega((H^*)^u_{>j})$ and (19), $\sigma_i(\lambda+\rho+\omega(H^u)) = \lambda+\rho+\omega((H^*)^u)$; thus

$$a_{\lambda+\rho+\omega(H^u)}(x\,|\,y) = -a_{\lambda+\rho+\omega((H^*)^u)}(x\,|\,y). \tag{21}$$

By (20) and (21), the contributions to (14) of two Bad Guys paired under the involution $H \mapsto H^*$ are negatives, and thus cancel. If a Bad Guy is paired with itself under $H \mapsto H^*$, then (21) implies that its contribution to (14) is 0. □

9. Bijection with Trapezoid Puzzles

In this section we define trapezoid puzzles, which are generalizations of Knutson-Tao puzzles. We give a weight-preserving bijection between equivariant Littlewood-Richardson skew tableaux and trapezoid puzzles which restricts to a bijection between positive equivariant Littlewood-Richardson skew tableaux and Knutson-Tao puzzles. Thus the formulas described in Section 2 for the structure constants $c^\nu_{\lambda,\mu}$ and $C^\nu_{\lambda,\mu}$, $\lambda,\mu,\nu \in \mathcal{P}_{d,n}$, can be indexed by trapezoid puzzles instead of equivariant Littlewood-Richardson skew tableaux.

9.1. **Trapezoid Puzzles.** A **puzzle piece** is one of the eight figures shown in Figure 3, each of whose edges has length 1 unit. Each puzzle piece is either an equilateral triangles or a rhombus, together with a fixed orientation, and a labelling of each edge with either a 1 or 0. The rightmost puzzle piece in Figure 3 is called an **equivariant puzzle piece**; we color it cyan.

Consider the isosceles trapezoid formed by placing an equilateral triangle of side length n on top of a rhombus of side length n and removing the common segment (see Figure 4, in which the common segment is darkened). Consider a partitioning P of this trapezoid into puzzle pieces in such a way that if two puzzle

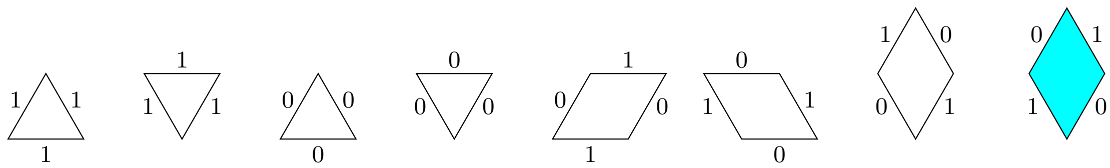


FIGURE 3. The eight puzzle pieces

pieces share an edge, then both puzzle pieces must have the same label on that edge. In this partitioning, one assumes that the common segment between the triangle and rhombus is not present. The **boundary** of P, denoted by ∂P, is the set of edges of P lying on the boundary of the trapezoid. It is divided into five parts: northeast, northwest, east, west, and south (denoted by ∂P_{NE}, ∂P_{NW}, ∂P_{E}, ∂P_{W}, and ∂P_{S}). These correspond to the northeast and northwest boundaries of the equilateral triangle, and the east, west, and south boundaries of the rhombus respectively. The partitioning P is called a **trapezoid puzzle** if ∂P_{E} and ∂P_{W} consist entirely of 0 edges. A **Knutson-Tao puzzle** is a trapezoid puzzle which has no 1-triangle puzzle pieces in the rhombus region.

One forms three n-digit binary words by reading the labels along ∂P_{NE}, ∂P_{NW}, and ∂P_{S}: the labels of ∂P_{NE} are read from top to bottom, the labels of ∂P_{NW} from bottom to top, and the labels of ∂P_{S} from left to right. To these three binary words we associate three partitions of $\mathcal{P}_{d,n}$ under the map $w \mapsto (\eta_1, \ldots, \eta_d) \in \mathcal{P}_{d,n}$, where η_j is the number of zeros of w which lie to the right of the j-th one of w from the left (for example, $0110001010 \mapsto (5,5,2,1) \in \mathcal{P}_{4,10}$). Denote by $\mathcal{P}^{\nu}_{\lambda,\mu}$ (resp. $\mathcal{P}^{\nu\,+}_{\lambda,\mu}$) the set of all trapezoid puzzles (resp. Knutson-Tao puzzles) P for which these three partitions are λ, μ, and ν, in that order.

Let D denote the common segment forming the south border of the triangle and the north border of the rhombus. For any equivariant puzzle piece of P, draw two lines from the center of the puzzle piece to D: one line L_1 parallel to ∂P_{NW} and the other L_2 parallel to ∂P_{NE}. The lines L_1 and L_2 cross D at $e - .5$ and $f - .5$ units from its right endpoint, respectively (e, f are both integers). If the equivariant puzzle piece lies above D, then $e > f$; if it lies below D, then $e < f$; if it is bisected by D, then $e = f$. The **factorial weight of the puzzle piece** is $y_e - y_f$, and the **equivariant weight of the puzzle piece** is $Y_{n+1-f} - Y_{n+1-e}$. Let c_P (resp. C_P) denote the product of the factorial weights (resp. equivariant weights) of all the equivariant puzzle pieces of P. For example, in Figure 4, $c_P = (y_5 - y_6)(y_5 - y_5)(y_5 - y_4)(y_2 - y_1) = 0$ and $C_P = (Y_4 - Y_3)(Y_4 - Y_4)(Y_4 - Y_5)(Y_7 - Y_8) = 0$.

Proposition 9.2. *There is a weight preserving bijection* $\Phi : \mathcal{P}^{\nu}_{\lambda,\mu} \to \mathcal{LR}^{\nu}_{\lambda,\mu}$*, which restricts to a weight preserving bijection* $\mathcal{P}^{\nu\,+}_{\lambda,\mu} \to \mathcal{LR}^{\nu\,+}_{\lambda,\mu}$*. By weight-preserving, we mean that for* $P \in \mathcal{P}^{\nu}_{\lambda,\mu}$*,* c_P *and* $c_{\Phi(P)}$ *are equal, and moreover are identical expressions; and similarly for* C_P *and* $C_{\Phi(P)}$*.*

Proof. The bijection Φ, illustrated in Figure 6, generalizes Tao's 'proof without words' of the bijection between puzzles and tableaux in the nonequivariant setting [V, Figure 11]. The object in the center of Figure 6 represents a truncated generic trapezoid puzzle P. The bottom of the rhombus portion of the trapezoid has been removed. To retrieve this portion, one extends the rhombus portion downward, meanwhile extending the white paths to the bottom of the figure. In the diagram,

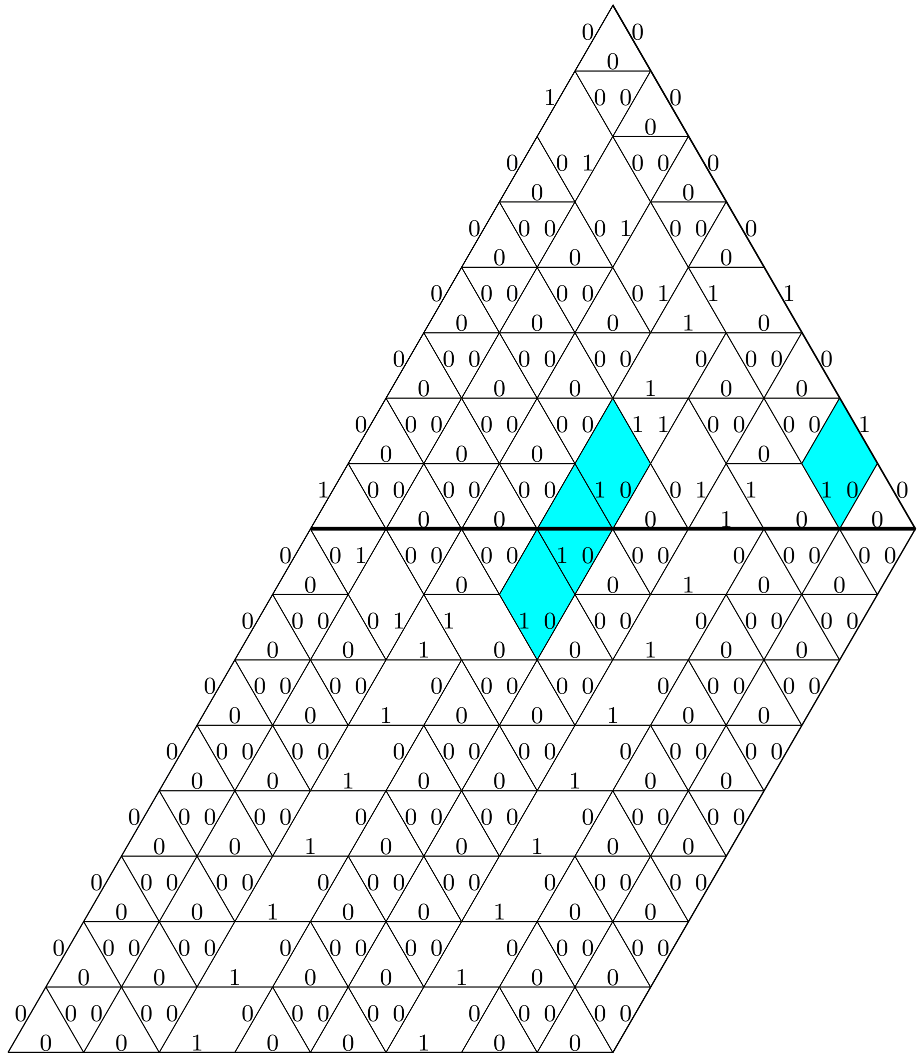


Figure 4. A trapezoid puzzle P, with $n = 8$, $d = 2$. The n-digit binary words of the NE, NW, and S sides of the boundary are 00001010, 10000010, and 00100100 respectively. Thus $P \in \mathcal{P}^{\nu+}_{\lambda,\mu}$, where $\lambda = (2, 1)$, $\mu = (6, 1)$, and $\nu = (4, 2)$. The darkened common segment is displayed only to illustrate that the trapezoidal shape is formed from an equilateral triangle and a rhombus; it is not part of the trapezoid puzzle.

black represents regions of 1 triangles, green represents regions of 0 triangles, white represents regions of non-equivariant rhombi, and cyan represents regions of equivariant rhombi.

From P, one may construct the Young tableau Y and reverse barred tableau B appearing in Figure 6. The shape of Y is determined by the lengths indicated on ∂P_{NE}, and the i-th row is filled with unbarred i's. The reverse barred tableau B is

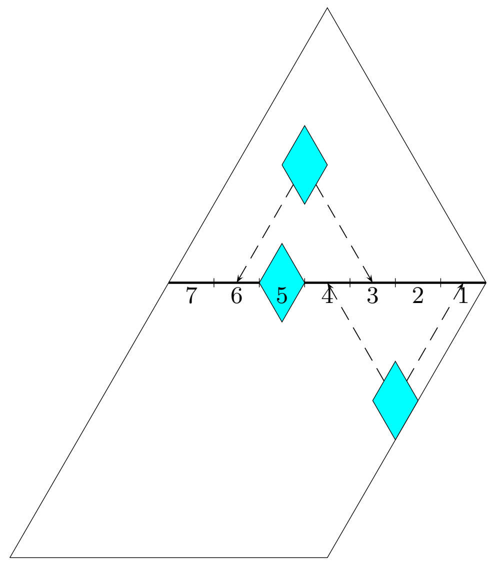


FIGURE 5. The three equivariant puzzle pieces have factorial weights $y_6 - y_3$, $y_5 - y_5$ $(= 0)$, and $y_1 - y_4$, and equivariant weights $Y_5 - Y_2$, $Y_3 - Y_3$ $(= 0)$, and $Y_7 - Y_4$ respectively.

constructed using the regions of P consisting of rhombus puzzle pieces labelled by $x_{i,j}$ (where $x = a, b, c,$ or d). To each puzzle piece in such a region there corresponds an entry of value j in row i of B. An equivariant puzzle piece corresponds to a barred entry of B; a non-equivariant puzzle piece corresponds to an unbarred entry of B. The skew barred tableau $\Phi(P)$ is constructed by placing Y above and to the right of B.

We list two properties of any $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$:

(a) $L|_{\mu}$ is column strict; and
(b) the unbarred column word of L is Yamanouchi.

Let $P \in \mathcal{P}^{\nu}_{\lambda,\mu}$. For $i \in \{1, \ldots, d\}$ (where $d = 4$ in Figure 6), there is a path P_i in P consisting of 1-triangles and rhombi which begins on ∂P_{NE}, moves only west or southwest, and ends on ∂P_{S} (see Figure 7). Each path P_i has **segments** $P_{i,j}$ consisting of the rhombus pieces lying in the regions of Figure 6 labelled by $x_{i,j}$ (where $x = a, b, c$, or d).

We list two properties of P:

(a)' for $i = 2, \ldots, d$ and all j, the distance from the leftmost edge of $P_{i,j}$ to ∂P_{NE} is greater than or equal to the distance from the leftmost edge of $P_{i-1,j}$ to to ∂P_{NE};
(b)' the interiors of the P_i do not touch; and

Properties (a)' and (b)' of P imply properties (a) and (b) of $\Phi(P)$ respectively. Conversely, given any $L \in \mathcal{LR}^{\nu}_{\lambda,\mu}$, Figure 6 shows how to construct a puzzle $\Phi^{-1}(L)$. Properties (a) and (b) of L ensure that the puzzle $\Phi^{-1}(L)$ can be constructed, and imply that it satisfies (a)' and (b)'. Uniqueness is clear.

To each equivariant puzzle piece of P there corresponds a barred entry of $\Phi(P)$, and they both determine the same factor $y_i - y_j$ of c_P and $c_{\Phi(P)}$ respectively. Therefore Φ is weight preserving, and thus restricts to a bijection from $\{P \in \mathcal{P}^{\nu}_{\lambda,\mu} \mid c_P > 0\}$ to $\{L \in \mathcal{LR}^{\nu}_{\lambda,\mu} \mid c_L > 0\} = \mathcal{LR}^{\nu\,+}_{\lambda,\mu}$. To see that the former set is $\mathcal{P}^{\nu\,+}_{\lambda,\mu}$, observe that P is not a Knutson-Tao puzzle if and only if P contains a 1 triangle lying below D if and only if P contains an equivariant puzzle piece which is bisected by D if and only if $c_P = 0$. $\square$

Using Theorem 2.4, Corollary 2.9, and Proposition 9.2, we obtain a proof of the following theorem; these formulas, expressed in terms of $\mathcal{P}^{\nu\,+}_{\lambda,\mu}$, are due to Knutson and Tao [KT].

Theorem 9.3. *For* $\lambda, \mu, \nu \in \mathcal{P}_{d,n}$,
$c^{\nu}_{\lambda,\mu} = \sum_{P \in \mathcal{P}^{\nu}_{\lambda,\mu}} c_P = \sum_{P \in \mathcal{P}^{\nu\,+}_{\lambda,\mu}} c_P$ *and* $C^{\nu}_{\lambda,\mu} = \sum_{P \in \mathcal{P}^{\nu}_{\lambda,\mu}} C_P = \sum_{P \in \mathcal{P}^{\nu\,+}_{\lambda,\mu}} C_P$.

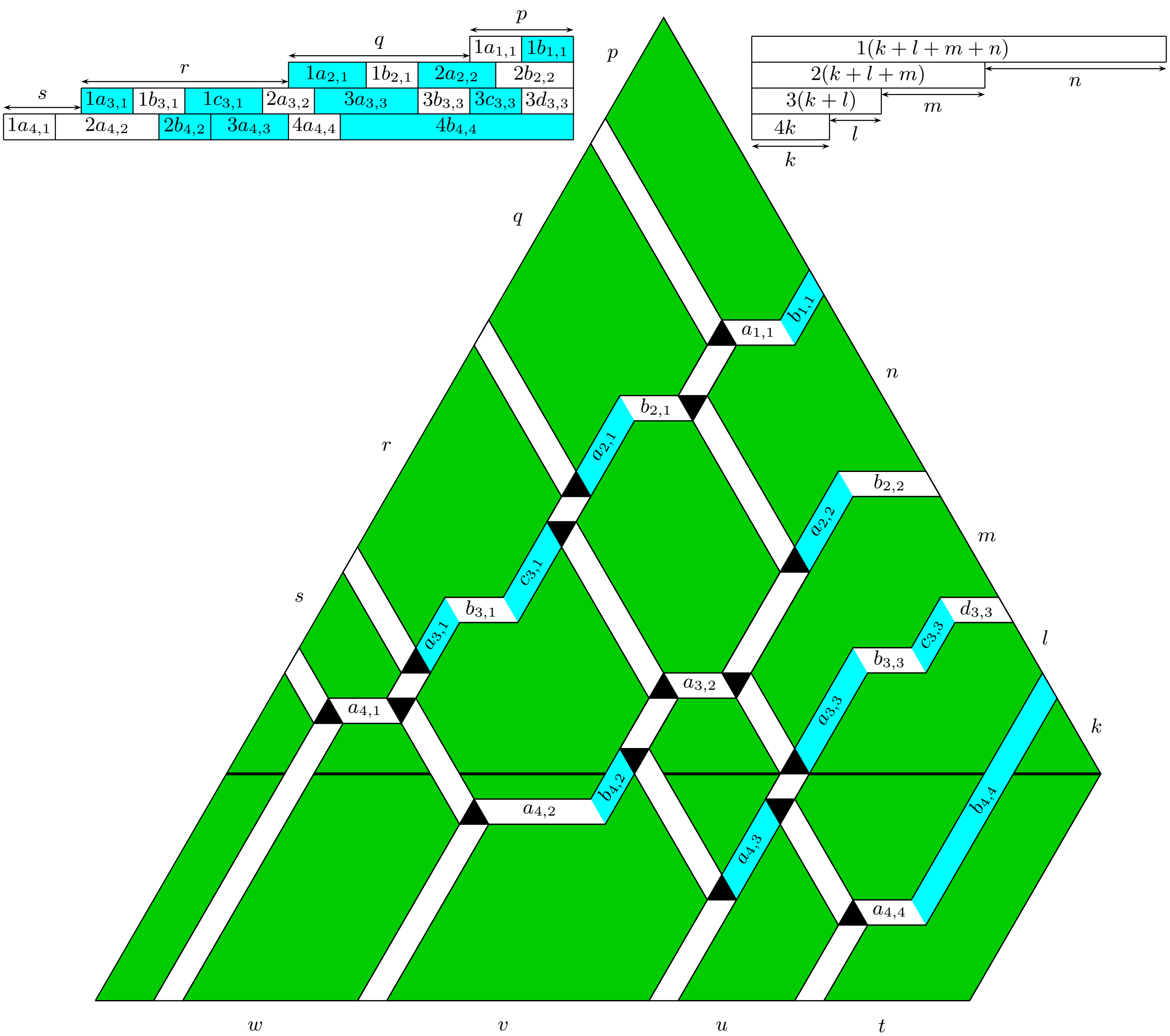


FIGURE 6. A generic trapezoid puzzle P (center), and its associated positive equivariant Littlewood-Richardson skew tableau $\Phi(P)$ (top-right, top-left). The bottom portion of P has been truncated. In P, black represents regions of 1 triangles, green represents regions of 0 triangles, white represents regions of non-equivariant rhombi, and cyan represents regions of equivariant rhombi.

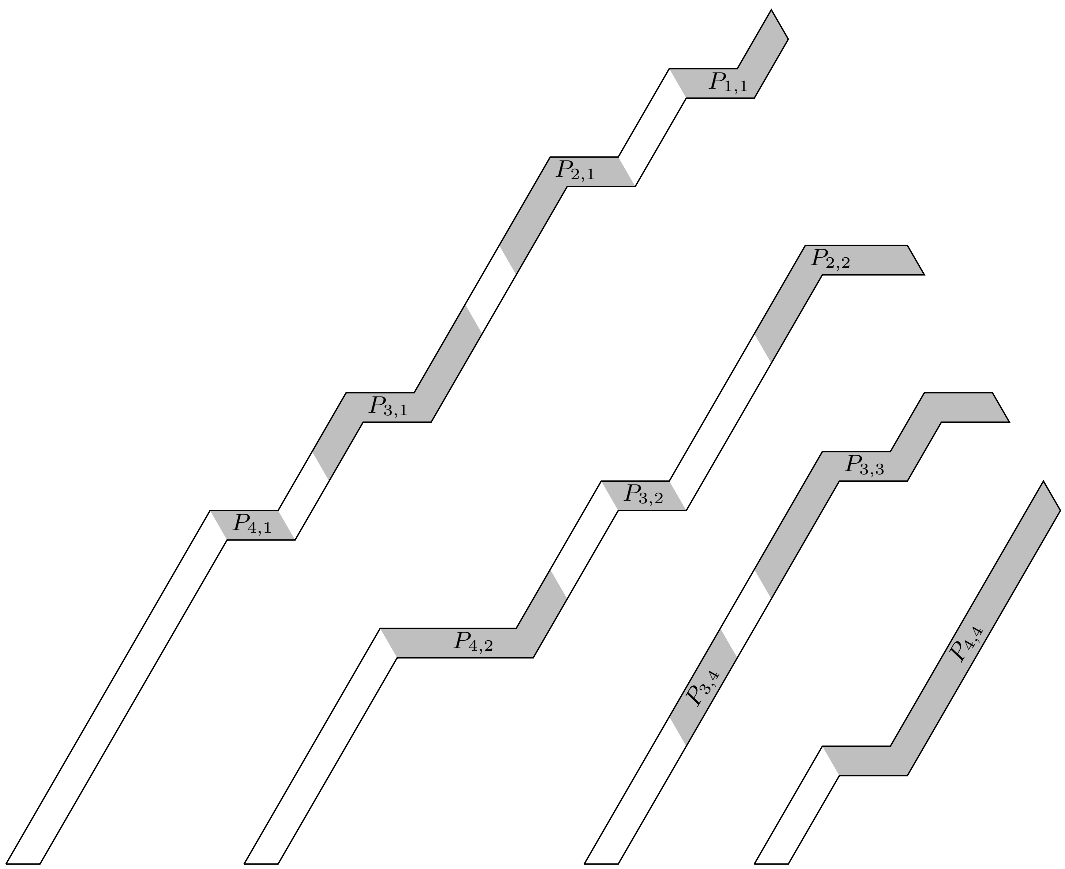


FIGURE 7. The paths P_i, $i = 1, \ldots, 4$, of the trapezoid puzzle P of Figure 6. The segments $P_{i,j}$ of each path are shaded. The segments may contain two types of puzzle pieces: equivariant puzzle pieces and rhombi with horizontal 0-edges.

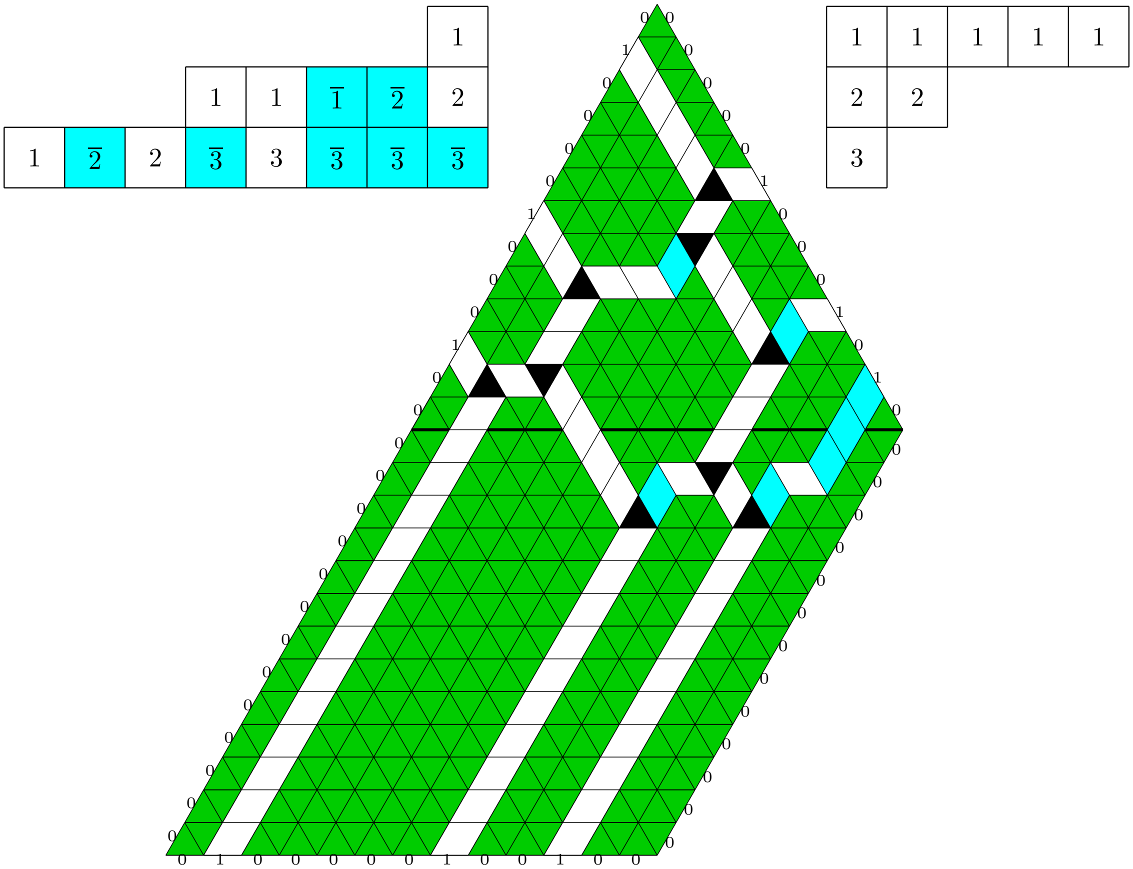


Figure 8. A trapezoid puzzle P (center), with the color scheme described in Figure 6. Here $d = 3$, $n = 13$, $\lambda = (5, 2, 1)$, $\mu = (8, 5, 1)$, $\nu = (9, 4, 2)$. The corresponding equivariant Littlewood-Richardson skew tableau $\Phi(P)$ appears top-right, top-left. The common segment D separating the triangle from the rhombus is darkened. The fact that 1-triangles lie below D implies that $c_P = C_P = 0$. Indeed, $c_P = (y_9 - y_4)(y_5 - y_2)(y_2 - y_1)(y_2 - y_2)(y_2 - y_3)(y_3 - y_5)(y_6 - y_8) = 0$, and $C_P = (Y_{10} - Y_5)(Y_{12} - Y_9)(Y_{13} - Y_{12})(Y_{12} - Y_{12})(Y_{11} - Y_{12})(Y_9 - Y_{11})(Y_6 - Y_8) = 0$.

10. The Molev-Sagan Rule

In this section we recall the Molev-Sagan rule for computing the coefficients $c_{\lambda,\mu}^{\nu}$ [MS].

The **forward skew diagram** $\lambda \star \mu$ is obtained by placing the Young diagram λ above and to the right of the Young diagram μ (see Figure 9). Note the difference between this object and the skew diagram $\lambda * \mu$ defined in Section 2: in $\lambda \star \mu$, μ is represented by a Young diagram rather than a reverse Young diagram. One defines **forward skew barred tableau** L, **unbarred column word of** L, L^u, and $L^u_{<a}$ in essentially the same way as their counterparts in Section 2.2, with the only difference being that the definitions are applied to the shape $\lambda \star \mu$ rather than $\lambda * \mu$.

Definition 10.1. *An* ***equivariant Littlewood-Richardson forward skew tableau*** *is a forward skew barred tableau whose unbarred column word is Yamanouchi. Denote the set of all equivariant Littlewood-Richardson forward skew tableaux of shape* $\lambda \star \mu$ *and unbarred content* ν *by* $\mathcal{LRF}^{\nu}_{\lambda,\mu}$.

For L a forward skew barred tableau, define

$$e_L = \prod_{\substack{a \in L \\ a \text{ barred}}} \left(y_{|a|' + \omega(L^u_{<a})_{|a|}} - y_{|a| + c(a) - r(a)} \right),$$

where the rows of μ are numbered from top to bottom and the columns from left to right, and $|a|' = d + 1 - |a|$.

Theorem 10.2 (Molev-Sagan Rule)**.** $c^{\nu}_{\lambda,\mu} = \displaystyle\sum_{L \in \mathcal{LRF}^{\nu}_{\lambda,\mu}} e_L,$

We remark that [MS, (8) and Theorem 3.1] is more general than Theorem 10.2, as it allows for the y variables in $s_\lambda(x \,|\, y)$ and the y variables in $s_\mu(x \,|\, y)$ of (1) to be two different families of variables.

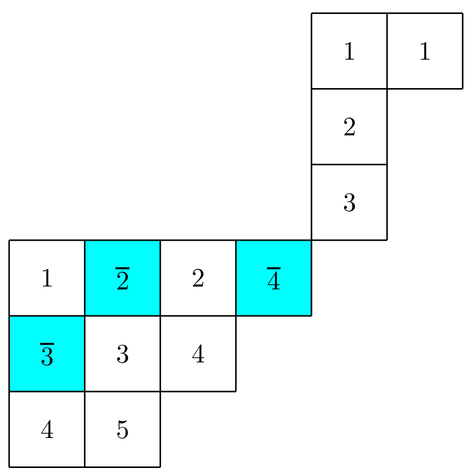
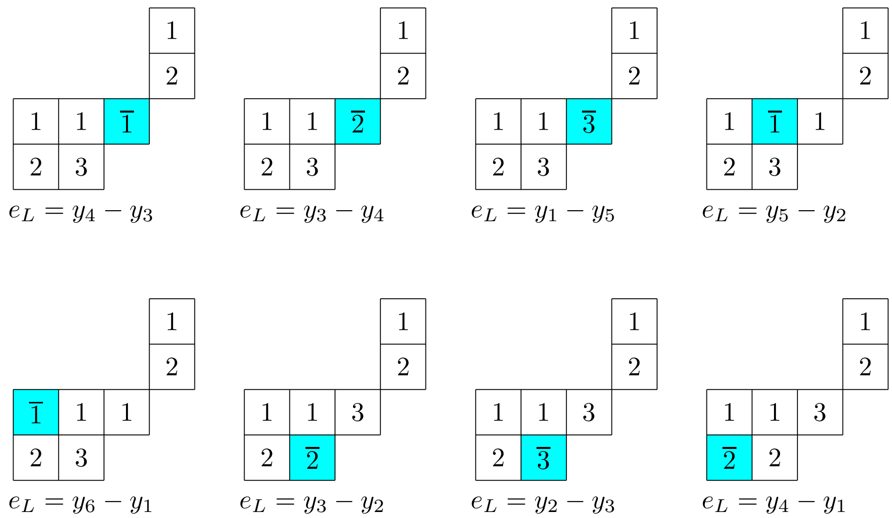


FIGURE 9. An equivariant Littlewood-Richardson forward skew tableau L of shape $\lambda \star \mu$ and unbarred content ν, where $\lambda = (2, 1, 1)$, $\mu = (4, 3, 2)$, and $\nu = (3, 2, 2, 2, 1)$. The unbarred column word, $1, 1, 2, 3, 2, 4, 3, 5, 1, 4$, is Yamanouchi. If $d = 5$, then $e_L = (y_2 - y_7)(y_6 - y_3)(y_5 - y_2)$.

In the following example, we recompute the coefficients $c^{\nu}_{\lambda,\mu}$ of Example 2.11 using the Molev-Sagan rule.

Example 10.3. *Let* $d = 3$, $\lambda = (1, 1)$, $\mu = (3, 2)$, *and* $\nu = (3, 2, 1)$. *We list all* $L \in \mathcal{LRF}^{\nu}_{\lambda,\mu}$, *and for each* L *we give* e_L*:*



By Theorem 10.2, $c^{\nu}_{\lambda,\mu} = (y_4 - y_3) + (y_3 - y_4) + (y_1 - y_5) + (y_5 - y_2) + (y_6 - y_1) + (y_3 - y_2) + (y_2 - y_3) + (y_4 - y_1)$.

The presence of terms of the form $y_i - y_j$ with both $i > j$ and $i < j$ in this example and the example of Figure 9 illustrate that in general the Molev-Sagan rule does not yield a positive formula for $c^{\nu}_{\lambda,\mu}$.

We remark that Corollary 2.8 does not hold if c_L is replaced by e_L and $\mathcal{LR}^{\nu}_{\lambda,\mu}$ by $\mathcal{LRF}^{\nu}_{\lambda,\mu}$. (In fact, Figure 9 gives a counterexample, since $\mu \not\subseteq \nu$, implying $c^{\nu}_{\lambda,\mu} = 0$, but $e_L \neq 0$.) This is due to the nonpositivity of the Molev-Sagan rule.

The Molev-Sagan rule and Theorem 2.4 of course produce the same coefficients $c^{\nu}_{\lambda,\mu}$. It would be interesting to deduce one rule directly from the other.

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Department of Mathematics, University of Georgia, Athens, GA 30602 USA
E-mail address: vkreiman@math.uga.edu