

Schubert Classes in the Equivariant K-Theory and Equivariant Cohomology of the Grassmannian

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Abstract

We give positive formulas for the restriction of a Schubert Class to a T -fixed point in the equivariant K-theory and equivariant cohomology of the Grassmannian. Our formulas rely on a result of Kodiyalam-Raghavan and Kreiman-Lakshmibai, which gives an equivariant Gröbner degeneration of a Schubert variety in the neighborhood of a T -fixed point of the Grassmannian.

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1 Introduction

The group T of diagonal matrices in $GL_n(\mathbb{C})$ acts on the Grassmannian $Gr_{d,n}$, with fixed point set indexed by $I_{d,n}$, the d element subsets of $\{1, \dots, n\}$. For $\beta \in I_{d,n}$, denote the corresponding T -fixed point by e_β . The T -equivariant

embedding $e_\beta \xrightarrow{i} Gr_{d,n}$ induces restriction homomorphisms i_K^* in T -equivariant K-theory and i_H^* in T -equivariant cohomology:

$$\begin{aligned} K_T^*(Gr_{d,n}) &\xrightarrow{i_K^*} K_T^*(e_\beta) \cong R(T) = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \\ H_T^*(Gr_{d,n}) &\xrightarrow{i_H^*} H_T^*(e_\beta) \cong \mathbb{C}[\mathfrak{t}^*] = \mathbb{C}[t_1, \dots, t_n] \end{aligned}$$

where $R(T)$ is the representation ring of T and \mathfrak{t} is the Lie algebra of T . The image of an element z of $K_T^*(Gr_{d,n})$ or $H_T^*(Gr_{d,n})$ under restriction to e_β is denoted by $z|_{e_\beta}$. The product maps

$$\begin{aligned} K_T^*(Gr_{d,n}) &\rightarrow \prod_{\beta \in I_{d,n}} K_T^*(e_\beta), \quad z \mapsto \prod_{\beta \in I_{d,n}} z|_{e_\beta} \\ H_T^*(Gr_{d,n}) &\rightarrow \prod_{\beta \in I_{d,n}} H_T^*(e_\beta), \quad z \mapsto \prod_{\beta \in I_{d,n}} z|_{e_\beta} \end{aligned}$$

are both injective. Thus, an element of $K_T^*(Gr_{d,n})$ or $H_T^*(Gr_{d,n})$ is determined by its restrictions to all e_β , $\beta \in I_{d,n}$.

The Schubert varieties of the Grassmannian are in bijection with the T -fixed points, and thus are also indexed by $I_{d,n}$. For $\alpha \in I_{d,n}$, denote the Schubert variety by X_α , and the corresponding Schubert classes in $K_T^*(Gr_{d,n})$ and $H_T^*(Gr_{d,n})$ by $[X_\alpha]_K$ and $[X_\alpha]_H$ respectively. In this paper, we obtain positive formulas for $[X_\alpha]_K|_{e_\beta}$ and $[X_\alpha]_H|_{e_\beta}$; the former formula implies the latter. Our formula for $[X_\alpha]_H|_{e_\beta}$ is positive in the sense of Graham [Gr01], and is expressed in terms of three different equivalent indexing sets: *families of nonintersecting paths in Young diagrams*, *subsets of Young diagrams*, and *Young tableaux*. Our formula for $[X_\alpha]_K|_{e_\beta}$ is positive in the sense of Griffith-Ram [GR04b], and is expressed in terms of *set-valued tableaux*, which are obtained as unions of Young tableaux.

More generally, for G/B , where G is any complex reductive group, positive formulas for restrictions of Schubert classes in equivariant cohomology were obtained by Billey [Bi99] and Andersen-Jantzen-Soergel [AJS94], and positive formulas for restrictions of Schubert classes in equivariant K-theory were obtained by Graham [Gr02], Willems [Wi06], and Knutson [Kn07]. Our combinatorial formulas can be deduced directly from these. Indeed, using the formula by Billey and Andersen-Jantzen-Soergel, Ikeda-Naruse [IN] independently obtained the same restriction formula as ours in the equivariant cohomology of the Grassmannian. The Ikeda-Naruse formula is expressed in terms of *excited Young diagrams*, which are the same as our subsets of Young diagrams. Ikeda-Naruse [IN] also obtained similar formulas for the symplectic and orthogonal Grassmannians (see also Ikeda [Ik]).

This paper should be viewed as a continuation of Kodiyalam-Raghavan [KR03], Kreiman-Lakshmibai [KL03], and Kreiman [Kre05]. In particular, our proofs and formulas rely on results of these papers. The main features of this approach are a geometric interpretation for the restriction formula in equivariant K-theory, in terms of intersections of coordinate spaces resulting from a Gröbner degeneration of local patches in the Grassmannian; the demonstration

of the equivalence of the nonintersecting path family model used by [KR03], [KL03], and [Kre05] (and first appearing in [Kra01, Kra05]), to other equivalent combinatorial models, which are in some senses simpler and are similar to other models in the literature; and the geometric interpretation of set-valued tableaux in the K-theory formula: set-valued tableaux are formed by taking ‘unions’ of ordinary Young tableaux, which corresponds geometrically to taking intersections of coordinate spaces of the Gröbner degeneration. The methods and results of this paper have been extended to the case of Symplectic Grassmannians by Kreiman [Kre06], by using a Gröbner degeneration of Ghorpade-Raghavan [GR04a]. Lakshmibai, Raghavan, and Sankaran [LRS06] used the results of [KR03], [KL03], and [Kre05] to give a determinantal formula for restrictions of Schubert classes in the equivariant cohomology of the Grassmannian.

There are other formulas for $[X_\alpha]_{\mathbb{K}|e_\beta}$ and $[X_\alpha]_{\mathbb{H}|e_\beta}$, most of which are non-positive. Letting $\mathcal{G}_\alpha(\mathbf{r}, \mathbf{t})$ and $\mathfrak{S}_\alpha(\mathbf{r}, \mathbf{t})$ denote the double Grothendieck and double Schubert polynomials [LS82a, LS82b] respectively for α , $[X_\alpha]_{\mathbb{K}|e_\beta} = \mathcal{G}_\alpha(\beta(\mathbf{t}), \mathbf{t})$ and $[X_\alpha]_{\mathbb{H}|e_\beta} = \mathfrak{S}_\alpha(\beta(\mathbf{t}), \mathbf{t})$, where on the right sides of both equations we view α and β as Grassmannian permutations rather than d -tuples. Letting $d_{\alpha,\beta}^\gamma$ and $c_{\alpha,\beta}^\gamma$ denote the linear structure constants for products of Schubert classes in the equivariant K-theory and equivariant cohomology respectively of the Grassmannian, $[X_\alpha]_{\mathbb{K}|e_\beta} = d_{\alpha,\beta}^\beta$ and $[X_\alpha]_{\mathbb{H}|e_\beta} = c_{\alpha,\beta}^\beta$. Hence the various formulas for $\mathcal{G}_\alpha(\mathbf{r}, \mathbf{t})$ and $\mathfrak{S}_\alpha(\mathbf{r}, \mathbf{t})$ (see [Ber92, BB93, Bi99, BKTY05, FK94, FK96, KMY05a, KMY05b, KM05, La90, LLT97, LS82a, LS82b, Le04, LRS04, Ma91], for example) and for $c_{\alpha,\beta}^\gamma$ (see [KT03, MS99]) and $d_{\alpha,\beta}^\gamma$ [Mc06] can be used to compute $[X_\alpha]_{\mathbb{K}|e_\beta}$ and $[X_\alpha]_{\mathbb{H}|e_\beta}$.

We now discuss our methods and results in greater detail.

Gröbner Degeneration

As mentioned above, our proof relies on a result of Kodiyalam-Raghavan [KR03], Kreiman-Lakshmibai [KL03], and Kreiman [Kre05], which gives an equivariant Gröbner degeneration of a local neighborhood of X_α centered at e_β to a reduced union $W_{\alpha,\beta}$ of coordinate subspaces W_1, \dots, W_q of W , an affine space whose coordinates are characters of T . Thus $[X_\alpha]_{\mathbb{H}|e_\beta} = \sum_{i=1}^q [W_i]_{\mathbb{H}}$, and by the inclusion-exclusion principle,

$$\begin{aligned} [X_\alpha]_{\mathbb{K}|e_\beta} &= [W_{\alpha,\beta}]_{\mathbb{K}} = \sum_{j=1}^q (-1)^{j+1} \sum_{1 \leq i_1 < \dots < i_j \leq q} [W_{i_1} \cap \dots \cap W_{i_j}]_{\mathbb{K}} \\ &= \sum_S N_S [W_S]_{\mathbb{K}}, \end{aligned}$$

where each W_S is an intersection $W_{i_1} \cap \dots \cap W_{i_j}$. The integer coefficient N_S accounts for the fact that W_S can in general be expressed as an intersection of W_i 's in more than one way. We show combinatorially that $N_S = 0, 1$, or -1 . Since W_S is itself a coordinate subspace, $[W_S]_{\mathbb{K}}$ is easily computed.

Gröbner degenerations are used to obtain the double Grothendieck and double Schubert polynomials for all permutations by Knutson and Miller [KM05],

and for vexillary permutations by Knutson, Miller, and Yong [KMY05a], [KMY05b]. In [KR03], [KL03], and [Kre05], only Grassmannians (i.e., Grassmannian permutations) are studied. Because attention is limited to the Grassmannian, it is possible for the authors to give an explicit Gröbner degeneration of local coordinate patches of the T -fixed points. In this paper, the local degeneration results in the positivity of the restriction formulas.

Combinatorial Models

As mentioned above, we show the equivalence of three combinatorial models: *families of nonintersecting paths on Young diagrams*, *subsets of Young diagrams*, and certain *Young tableaux*. These objects index the subspaces W_i , $i = 1, \dots, q$. The families of nonintersecting paths appeared first in Krattenthaler [Kra01, Kra05] and subsequently in [KR03], [KL03], and [Kre05]. In this paper, we view the families of nonintersecting paths as living inside an appropriate Young diagram; the subsets of Young diagrams are simply the complementary boxes. Subsets of Young diagrams are in some respects simpler objects. They were discovered independently by Ikeda-Naruse [IN], and are quite similar to RC graphs or reduced pipe dreams [BB93, FK94, KM05] for Grassmannian permutations.

Although the three models are equally suitable for expressing our equivariant cohomology formula, we find the tableau model to be the simplest one for deriving and expressing the equivariant K-theory formula. Each $[W_i]_{\mathbb{K}}$ is naturally indexed by a semistandard Young tableau P_i , and each $[W_{i_1} \cap \dots \cap W_{i_j}]_{\mathbb{K}}$ is naturally indexed by the ‘union’ $S = P_{i_1} \cup \dots \cup P_{i_j}$, which is a set-valued tableau. The S for which $N_S \neq 0$ are precisely those which are semistandard.

Semistandard set-valued tableaux were introduced by Buch [Bu02], who used them to give a formula for the linear structure constants for products of Schubert classes in the K-theory of the Grassmannian. Buch also expressed Grothendieck polynomials for Grassmannian permutations in terms of semistandard set-valued tableaux. Knutson, Miller, and Yong [KMY05a, KMY05b] give several formulas for double Grothendieck and double Schubert polynomials for vexillary permutations in terms of flagged set-valued tableaux. McNamara [Mc06] describes *factorial Grothendieck polynomials*, and structure constants for their products, in terms of set-valued tableaux.

The paper is organized as follows. In Section 2, we state our formulas for $[X_{\alpha}]_{\mathbb{K}}|_{e_{\beta}}$ and $[X_{\alpha}]_{\mathbb{K}}|_{e_{\beta}}$. In Section 3, we present basic definitions and properties of equivariant K-theory and equivariant cohomology for affine spaces and affine varieties. In Section 4, we give the main arguments for the proof of our formulas (Proposition 2.2), omitting the proofs of two lemmas. In Sections 5 and 6, we prove the first of these two lemmas, by translating a result of [KR03], [KL03], and [Kre05] into the language of semistandard Young tableaux. In Section 7, we prove the second lemma, which computes N_S .

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2 Statement of Results

Let d and n be fixed positive integers, $0 < d < n$. Let $I_{d,n}$ be the set of d -element subsets of $\{1, \dots, n\}$, where we always assume the entries of such a subset are listed in increasing order. We define the **complement** of $\alpha = \{\alpha(1), \dots, \alpha(d)\} \in I_{d,n}$ by $\alpha' = \{1, \dots, n\} \setminus \alpha$ and the **length** of α by $l(\alpha) = (\alpha(1)-1) + \dots + (\alpha(d)-d)$. Let $J_{d,n}$ be the set of partitions $\lambda = (\lambda_1, \dots, \lambda_d)$ such that $n-d \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0$. There is a standard bijection $\pi : I_{d,n} \rightarrow J_{d,n}$ given by $\pi(\{\alpha(1), \dots, \alpha(d)\}) = (\alpha(d)-d, \dots, \alpha(1)-1)$. Let $\alpha, \beta \in I_{d,n}$ be fixed.

The **Grassmannian** $Gr_{d,n}$ is the set of all d -dimensional complex subspaces of \mathbb{C}^n . Let $\{e_1, \dots, e_n\}$ be the standard basis for \mathbb{C}^n . Define $e_\alpha = \text{Span}\{e_{\alpha(1)}, \dots, e_{\alpha(d)}\} \in Gr_{d,n}$. Consider the opposite standard flag, whose i -th space is $F_i = \text{Span}\{e_n, \dots, e_{n-i+1}\}$, $i = 1, \dots, n$. The **Schubert variety** X_α (which is sometimes called an *opposite* Schubert variety) is defined by incidence relations:

$$X_\alpha = \{V \in Gr_{d,n} \mid \dim(V \cap F_i) \geq \dim(e_\alpha \cap F_i), i = 1, \dots, n\}.$$

A **Young diagram** is a collection of boxes arranged into a left and top justified array. If the i -th row of a diagram has λ_i boxes, $i = 1, \dots, r$, then we say that the **shape** of the diagram is the partition $\lambda = (\lambda_1, \dots, \lambda_r)$. A **set-valued tableau** is an assignment of a nonempty set of positive integers to each box of a diagram. The entries of a set-valued tableau S are the positive integers in the boxes. If a positive integer occurs in more than one box of S , then we consider the separate occurrences of the positive integer to be distinct entries of S . A **Young tableau** is a special type of set-valued tableau in which each box contains a single entry.

A set-valued tableau is said to be **semistandard** if all entries of any box B are less than or equal to all entries of the box to the right of B and strictly less than all entries of the box below B .

1	2, 3	3	4, 6, 7	7, 9
2	4, 5, 7	8, 9		
4, 6	8			

Figure 1: A semistandard set-valued tableau

If $\mu = (\mu_1, \dots, \mu_h)$ is any partition, then a set-valued tableau S is said to be **on** μ if, for every entry x of S , $x \leq h$ and

$$x + c(x) - r(x) \leq \mu_x, \quad (1)$$

where $r(x)$ and $c(x)$ are the row and column numbers of the box containing x . Note that the condition $x \leq h$ is required for μ_x , and thus (1), to be well-defined.

Example 2.1. Let $\lambda = (2, 1)$, $\mu = (4, 4, 2, 1)$. The following list gives all semistandard set-valued tableaux on μ of shape λ :

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Denote the set of semistandard set-valued tableaux on μ of shape λ by $\text{SSVT}_{\lambda, \mu}$ and the set of semistandard Young tableaux on μ of shape λ by $\text{SSYT}_{\lambda, \mu}$.

Proposition 2.2. Let $\lambda = \pi(\alpha)$, $\mu = \pi(\beta)$. Then

$$(i) [X_\alpha]_{\kappa|e_\beta} = (-1)^{l(\alpha)} \sum_{S \in \text{SSVT}_{\lambda, \mu}} \prod_{x \in S} \left(\frac{t_{\beta(d+1-x)}}{t_{\beta'(x+c(x)-r(x))}} - 1 \right).$$

$$(ii) [X_\alpha]_{\mathcal{H}|e_\beta} = \sum_{S \in \text{SSYT}_{\lambda, \mu}} \prod_{x \in S} (t_{\beta(d+1-x)} - t_{\beta'(x+c(x)-r(x))}).$$

Example 2.3. Consider $Gr_{3,6}$, $\alpha = \{1, 3, 5\}$, $\beta = \{2, 5, 6\}$. Then $l(\alpha) = 3$, $\pi(\alpha) = (2, 1, 0)$, $\pi(\beta) = (3, 3, 1)$. The semistandard set-valued tableaux on $\pi(\beta)$ of shape $\pi(\alpha)$ are:

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Therefore,

$$[X_\alpha]_{\kappa|e_\beta} = - \left[\left(\frac{t_6}{t_1} - 1 \right) \left(\frac{t_6}{t_3} - 1 \right) \left(\frac{t_5}{t_1} - 1 \right) + \left(\frac{t_6}{t_1} - 1 \right) \left(\frac{t_4}{t_2} - 1 \right) \left(\frac{t_5}{t_1} - 1 \right) \right. \\ \left. + \left(\frac{t_6}{t_1} - 1 \right) \left(\frac{t_6}{t_3} - 1 \right) \left(\frac{t_4}{t_2} - 1 \right) \left(\frac{t_5}{t_1} - 1 \right) \right].$$

$$[X_\alpha]_{\mathcal{H}|e_\beta} = (t_6 - t_1)(t_6 - t_3)(t_5 - t_1) + (t_6 - t_1)(t_4 - t_2)(t_5 - t_1).$$

Remark 2.4. In Proposition 2.2(i), the condition that S is on μ implies that each term in the product is of the form $t_b/t_a - 1$, $b > a$, and in Proposition 2.2(ii), the condition that S is on μ implies that each term in the product is of the form $t_b - t_a$, $b > a$. Indeed, for $\mu = \pi(\beta)$, one can show that $\mu_j = \#\{i \in \{1, \dots, n-d\} \mid \beta'(i) < \beta(d+1-j)\}$, $j = 1, \dots, d$. Therefore $i \leq \mu_j \iff \beta'(i) < \beta(d+1-j)$. Substituting $i = x + c(x) - r(x)$, $j = x$, we obtain: (1) $\iff x + c(x) - r(x) \leq \mu_x \iff \beta'(x + c(x) - r(x)) < \beta(d+1-x)$.

3 Equivariant K-Theory in Affine Spaces

The equivariant K-theory $K_T^*(V)$ of an algebraic variety V with a T action is defined to be the Grothendieck group of equivariant coherent sheaves of \mathcal{O}_V modules. If $Y \subset V$ is a T -stable closed subvariety, then we define $[Y]_{\mathbf{K}}$ to be the class of the structure sheaf \mathcal{O}_Y of Y .

In this section we assume that V is the affine space \mathbb{C}^m . In this case, the notion of coherent sheaves of \mathcal{O}_V modules can be replaced by that of finitely generated $\mathbb{C}[V]$ modules. If Y is a T -stable closed subvariety of V , then $[Y]_{\mathbf{K}}$ is just $[\mathbb{C}[Y]]_{\mathbf{K}}$. We also have that $K_T^*(V) \cong K_T^*(\mathbf{0})$, which can be identified with the representation ring of T , $R(T) = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

For any (possibly infinite dimensional) T -module L , define $\text{Char}(L) \in \mathbb{C}[[t_1^{\pm 1}, \dots, t_n^{\pm 1}]]$ to be the character of L under the T action (one also views $\text{Char}(L)$ as the \mathbb{Z}^n -graded Hilbert function of L , where each character of L is graded by its T -weight). For $\mathbf{d} \in \mathbb{Z}^n$, define $\mathbb{C}[V](-\mathbf{d})$ to be $\mathbb{C}[V]$ with modified T -action: the characters of $\mathbb{C}[V](-\mathbf{d})$ are the same as those of $\mathbb{C}[V]$, but with weight $\mathbf{t}^{\mathbf{d}}$ times greater. We use the standard identification

$$[\mathbb{C}[V](-\mathbf{d})]_{\mathbf{K}} = \mathbf{t}^{\mathbf{d}} = \frac{\text{Char}(\mathbb{C}[V](-\mathbf{d}))}{\text{Char}(\mathbb{C}[V])}.$$

Let Y be a T -stable closed subvariety of V . There is a free equivariant resolution

$$0 \rightarrow \mathcal{E}_r \rightarrow \dots \rightarrow \mathcal{E}_1 \rightarrow \mathbb{C}[Y] \rightarrow 0, \quad \text{where} \quad \mathcal{E}_i = \bigoplus_{j=1}^{u_i} \mathbb{C}[V](-\mathbf{d}_{ij}).$$

Since

$$[\mathcal{E}_i]_{\mathbf{K}} = \sum_{j=1}^{u_i} [\mathbb{C}[V](-\mathbf{d}_{ij})]_{\mathbf{K}} = \sum_{j=1}^{u_i} \frac{\text{Char}(\mathbb{C}[V](-\mathbf{d}_{ij}))}{\text{Char}(\mathbb{C}[V])} = \frac{\text{Char}(\mathcal{E}_i)}{\text{Char}(\mathbb{C}[V])},$$

it follows that

$$[Y]_{\mathbf{K}} = \sum_{i=1}^r (-1)^{i+1} [\mathcal{E}_i]_{\mathbf{K}} = \sum_{i=1}^r (-1)^{i+1} \frac{\text{Char}(\mathcal{E}_i)}{\text{Char}(\mathbb{C}[V])} = \frac{\text{Char}(\mathbb{C}[Y])}{\text{Char}(\mathbb{C}[V])}. \quad (2)$$

Example 3.1. Let $V = \mathbb{C}^3$, and let $y_1, y_2, y_3 \in \mathbb{C}[V]$ be the standard coordinate functions on V . Suppose that $T = (\mathbb{C}^*)^4$ acts on V , and hence on $\mathbb{C}[V]$, and suppose that for $\mathbf{t} = \text{diag}(t_1, t_2, t_3, t_4) \in T$,

$$\mathbf{t}(y_1) = \frac{t_4}{t_1} y_1, \quad \mathbf{t}(y_2) = \frac{t_2}{t_3} y_2, \quad \mathbf{t}(y_3) = t_1^{-2} y_3.$$

Then

$$\text{Char}(\mathbb{C}[V]) = \sum_{i,j,k=0}^{\infty} \left(\frac{t_4}{t_1}\right)^i \left(\frac{t_2}{t_3}\right)^j (t_1^{-2})^k = \frac{1}{(1 - \frac{t_4}{t_1})(1 - \frac{t_2}{t_3})(1 - t_1^{-2})}.$$

Let $Y \subset V$ be the y_1 -axis. Then

$$\text{Char}(\mathbb{C}[Y]) = \sum_{i=0}^{\infty} \left(\frac{t_4}{t_1}\right)^i = \frac{1}{(1 - \frac{t_4}{t_1})} = \frac{(1 - \frac{t_2}{t_3})(1 - t_1^{-2})}{(1 - \frac{t_4}{t_1})(1 - \frac{t_2}{t_3})(1 - t_1^{-2})}.$$

Therefore, by (2), $[Y]_{\mathcal{K}} = (1 - \frac{t_2}{t_3})(1 - t_1^{-2})$.

Let $y_1, \dots, y_m \in \mathbb{C}[V]$ be the standard coordinate functions on $V = \mathbb{C}^m$. We denote by $V(\{y_{j_1}, \dots, y_{j_k}\})$ the coordinate subspace of V defined by the vanishing of y_{j_1}, \dots, y_{j_k} .

Lemma 3.2. Let $\chi_i, i = 1, \dots, m$ be characters of T . Suppose that T acts on V , and hence on $\mathbb{C}[V]$, and suppose that $\mathbf{t}(y_i) = \chi_i(\mathbf{t})y_i, \mathbf{t} \in T, i = 1, \dots, m$.

(i) If $W \subset V$ is the coordinate subspace $W = V(\{y_{j_1}, \dots, y_{j_k}\})$, then

$$[W]_{\mathcal{K}} = (1 - \chi_{j_1}(\mathbf{t})) \cdots (1 - \chi_{j_k}(\mathbf{t})). \quad (3)$$

(ii) If $W \subset V$ is the union of coordinate subspaces W_1, \dots, W_q , then

$$[W]_{\mathcal{K}} = \sum_{j=1}^q (-1)^{j+1} \sum_{1 \leq i_1 < \dots < i_j \leq q} [W_{i_1} \cap \dots \cap W_{i_j}]_{\mathcal{K}}. \quad (4)$$

Proof. (i) is an easy generalization of Example 3.1.

(ii) By the inclusion-exclusion principle,

$$\begin{aligned} \text{Char}(W) &= \text{Char}(W_1 \cup \dots \cup W_q) \\ &= \sum_{j=1}^q (-1)^{j+1} \sum_{1 \leq i_1 < \dots < i_j \leq q} \text{Char}(W_{i_1} \cap \dots \cap W_{i_j}) \end{aligned}$$

The result now follows from (2). \square

Note that each $W_{i_1} \cap \dots \cap W_{i_j}$ in (4) is itself a coordinate subspace, so (3) can be used to compute its class.

4 The Class of an Opposite Schubert Variety

The **Plücker map** $\text{pl} : Gr_{d,n} \rightarrow \mathbb{P}(\wedge^d \mathbb{C}^n)$ is defined by $\text{pl}(W) = [w_1 \wedge \cdots \wedge w_d]$, where $\{w_1, \dots, w_d\}$ is any basis for W . It is well known that pl is a closed immersion. Thus $Gr_{d,n}$ inherits the structure of projective variety, as does $X_\alpha \subset Gr_{d,n}$.

Reduction to an Affine Variety

Under the Plücker map, e_β maps to $[e_{\beta_1} \wedge \cdots \wedge e_{\beta_d}] \in \mathbb{P}(\wedge^d \mathbb{C}^n)$. Define p_β to be homogeneous (**Plücker**) coordinate $[e_{\beta_1} \wedge \cdots \wedge e_{\beta_d}]^* \in \mathbb{C}[\mathbb{P}(\wedge^d \mathbb{C}^n)]$. Let \mathcal{O}_β be the distinguished open set of $Gr_{d,n}$ defined by $p_\beta \neq 0$. Then \mathcal{O}_β is isomorphic to the affine space $\mathbb{C}^{d(n-d)}$, with e_β the origin. Indeed, \mathcal{O}_β can be identified with the space of matrices in $M_{n \times d}$ in which rows β_1, \dots, β_d are the rows of the $d \times d$ identity matrix, and rows $\beta'_1, \dots, \beta'_{n-d}$ contain arbitrary elements of \mathbb{C} . Under this identification, the rows of \mathcal{O}_β are indexed by $\{1, \dots, n\}$, and the columns by β .

Example 4.1. Let $d = 3$, $n = 7$, $\beta = \{2, 5, 7\}$. Then $\beta' = \{1, 3, 4, 6\}$, and

$$\mathcal{O}_\beta = \left\{ \begin{pmatrix} y_{12} & y_{15} & y_{17} \\ 1 & 0 & 0 \\ y_{32} & y_{35} & y_{37} \\ y_{42} & y_{45} & y_{47} \\ 0 & 1 & 0 \\ y_{62} & y_{65} & y_{67} \\ 0 & 0 & 1 \end{pmatrix}, y_{ab} \in \mathbb{C} \right\}.$$

The space \mathcal{O}_β is T -stable, and for $\mathbf{t} = \text{diag}(t_1, \dots, t_n) \in T$ and coordinate functions $y_{ab} \in \mathbb{C}[\mathcal{O}_\beta]$,

$$\mathbf{t}(y_{ab}) = \frac{t_b}{t_a} y_{ab}. \quad (5)$$

The equivariant embeddings $e_\beta \xrightarrow{j} \mathcal{O}_\beta \xrightarrow{k} Gr_{d,n}$ induce homomorphisms

$$K_T^*(Gr_{d,n}) \xrightarrow{k^*} K_T^*(\mathcal{O}_\beta) \xrightarrow{j^*} K_T^*(e_\beta).$$

The map j^* is an isomorphism, identifying $K_T^*(\mathcal{O}_\beta)$ with $K_T^*(e_\beta)$. Define $Y_{\alpha,\beta} = X_\alpha \cap \mathcal{O}_\beta$. We have

$$[X_\alpha]_{\mathbb{K}}|_{e_\beta} = j^* \circ k^*([X_\alpha]_{\mathbb{K}}) = j^*([k^{-1}X_\alpha]_{\mathbb{K}}) = j^*([Y_{\alpha,\beta}]_{\mathbb{K}}) = [Y_{\alpha,\beta}]_{\mathbb{K}}. \quad (6)$$

Applying analogous arguments for equivariant cohomology, we obtain

$$[X_\alpha]_{\mathbb{H}}|_{e_\beta} = [Y_{\alpha,\beta}]_{\mathbb{H}}. \quad (7)$$

Reduction to a Union of Coordinate Subspaces

Let $\lambda = \pi(\alpha)$, $\mu = \pi(\beta)$. Let $\text{SVT}_{\lambda,\mu}$ denote the set of all set-valued tableaux (not necessarily semistandard) of shape λ on μ . For $S \in \text{SVT}_{\lambda,\mu}$, define

$$W_S = V(\{y_{\beta'(x+c(x)-r(x)),\beta(d+1-x)}, x \in S\}), \quad (8)$$

a coordinate subspace of \mathcal{O}_β . Define

$$W_{\alpha,\beta} = \bigcup_{P \in \text{SSYT}_{\lambda,\mu}} W_P. \quad (9)$$

The following lemma, whose proof appears in Section 6, reduces our problem to computing the class of a union of coordinate subspaces.

Lemma 4.2. $[Y_{\alpha,\beta}]_\kappa = [W_{\alpha,\beta}]_\kappa$

Let R and S be two set-valued tableaux of shape λ . Define the **union** $R \cup S$ to be the set-valued tableau of shape λ whose entries in each box are the unions of the entries of R and the entries of S in that box. If R and S are both on μ , then $R \cup S$ is on μ , and $W_{R \cup S} = W_R \cap W_S$. We say that R is **contained in** S , and write $R \subset S$, if each entry in each box of R is also an entry in the same box of S . In this case, if S is on μ , then R is on μ .

Let S be a semistandard tableau of shape λ . Define $\text{SSYT}(S)$ to be the set of semistandard Young tableaux of shape λ which are contained in S , and define $q_S = |\text{SSYT}(S)|$. Define $N_{S,j}$ to be the number of j element subsets of $\text{SSYT}(S)$ whose unions equal S , and define $N_S = \sum_{j=1}^{q_S} (-1)^{j+1} N_{S,j}$.

Lemma 4.3. $[W_{\alpha,\beta}]_\kappa = \sum_{S \in \text{SVT}_{\lambda,\mu}} N_S \prod_{x \in S} \left(1 - \frac{t_{\beta(d+1-x)}}{t_{\beta'(x+c(x)-r(x))}}\right)$.

Proof. Let P_1, \dots, P_q be an enumeration of $\text{SSYT}_{\lambda,\mu}$. Note that for any $S \in$

$\text{SVT}_{\lambda,\mu}$, $\text{SSYT}(S) \subset \text{SSYT}_{\lambda,\mu}$; thus $q_S \leq q$. By (9) and Lemma 3.2(ii),

$$\begin{aligned}
[W_{\alpha,\beta}]_{\mathbb{K}} &= [W_{P_1} \cup \cdots \cup W_{P_q}]_{\mathbb{K}} \\
&= \sum_{j=1}^q (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq q} [W_{P_{i_1}} \cap \cdots \cap W_{P_{i_j}}]_{\mathbb{K}} \\
&= \sum_{j=1}^q (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq q} [W_{P_{i_1} \cup \cdots \cup P_{i_j}}]_{\mathbb{K}} \\
&= \sum_{j=1}^q (-1)^{j+1} \sum_{S \in \text{SVT}_{\lambda,\mu}} \sum_{\substack{P_{i_1} \cup \cdots \cup P_{i_j} = S, \\ 1 \leq i_1 < \cdots < i_j \leq q}} [W_S]_{\mathbb{K}} \\
&= \sum_{j=1}^q (-1)^{j+1} \sum_{S \in \text{SVT}_{\lambda,\mu}} N_{S,j} [W_S]_{\mathbb{K}} \\
&= \sum_{S \in \text{SVT}_{\lambda,\mu}} \sum_{j=1}^q (-1)^{j+1} N_{S,j} [W_S]_{\mathbb{K}} \\
&= \sum_{S \in \text{SVT}_{\lambda,\mu}} \sum_{j=1}^{q_S} (-1)^{j+1} N_{S,j} [W_S]_{\mathbb{K}} \\
&= \sum_{S \in \text{SVT}_{\lambda,\mu}} N_S [W_S]_{\mathbb{K}}.
\end{aligned}$$

By (8), (5), and Lemma 3.2(i), for each $S \in \text{SVT}_{\lambda,\mu}$,

$$[W_S]_{\mathbb{K}} = \prod_{x \in S} \left(1 - \frac{t^{\beta(d+1-x)}}{t^{\beta'(x+c(x)-r(x))}} \right).$$

□

For set-valued tableau S , define $|S|$ (resp. $\|S\|$) to be the total number of entries (resp. boxes) of S . The proof of the following Lemma appears in Section 7.

Lemma 4.4. (i) If S is semistandard, then $N_S = (-1)^{|S|+\|S\|}$.
(ii) If S is not semistandard, then $N_S = 0$.

Proof of Proposition 2.2. (i) Combining the preceding reductions and lemmas:

$$\begin{aligned}
[X_\alpha]_{\mathbb{K}}|_{e_\beta} &= [Y_{\alpha,\beta}]_{\mathbb{K}} \\
&= [W_{\alpha,\beta}]_{\mathbb{K}} \\
&= \sum_{S \in \text{SVT}_{\lambda,\mu}} N_S \prod_{x \in S} \left(1 - \frac{t_{\beta(d+1-x)}}{t_{\beta'(x+c(x)-r(x))}} \right) \\
&= \sum_{S \in \text{SSVT}_{\lambda,\mu}} (-1)^{|S|+\|S\|} \prod_{x \in S} \left(1 - \frac{t_{\beta(d+1-x)}}{t_{\beta'(x+c(x)-r(x))}} \right) \\
&= (-1)^{l(\alpha)} \sum_{S \in \text{SSVT}_{\lambda,\mu}} (-1)^{|S|} \prod_{x \in S} \left(1 - \frac{t_{\beta(d+1-x)}}{t_{\beta'(x+c(x)-r(x))}} \right) \\
&= (-1)^{l(\alpha)} \sum_{S \in \text{SSVT}_{\lambda,\mu}} \prod_{x \in S} \left(\frac{t_{\beta(d+1-x)}}{t_{\beta'(x+c(x)-r(x))}} - 1 \right).
\end{aligned}$$

(ii) By (??) and [KM05], Lemma 1.1.4, $[X_{\alpha,\beta}]_{\mathbb{H}}|_{e_\beta}(\mathbf{t})$ equals the sum of the lowest degree terms of $[X_{\alpha,\beta}]_{\mathbb{K}}|_{e_\beta}(\mathbf{1} - \mathbf{t})$. One checks that the sum of the lowest degree terms of

$$[X_{\alpha,\beta}]_{\mathbb{K}}|_{e_\beta}(\mathbf{1} - \mathbf{t}) = (-1)^{l(\alpha)} \sum_{S \in \text{SSVT}_{\lambda,\mu}} \prod_{x \in S} \left(\frac{1 - t_{\beta(d+1-x)}}{1 - t_{\beta'(x+c(x)-r(x))}} - 1 \right)$$

equals

$$\sum_{S \in \text{SSYT}_{\lambda,\mu}} \prod_{x \in S} (t_{\beta(d+1-x)} - t_{\beta'(x+c(x)-r(x))}).$$

□

5 Families of Nonintersecting Paths on Young Diagrams

In this section, we introduce a set $\mathcal{F}_{\lambda,\mu}$ of families of nonintersecting paths on Young diagrams, which we then show to be identical to the set $\mathcal{F}_{\lambda,\mu}''$ of families of nonintersecting paths which appear in [KR03], [Kra01], [Kra05], [KL03], and [Kre05], despite the fact that $\mathcal{F}_{\lambda,\mu}$ and $\mathcal{F}_{\lambda,\mu}''$ are defined quite differently. Thus one can view the result of this section as giving an alternate way of defining or expressing the path families of [KR03], [Kra01], [Kra05], [KL03], and [Kre05]. In terms of this alternate definition, one can more easily see the equivalence between the path families and the other combinatorial models in Section 6.

Let μ be a partition and D_μ the corresponding Young diagram. We denote by (i, j) the box of D_μ at row i (from the top), column j (from the left). We impose the following order on boxes of D_μ : if $(i, j), (k, l) \in D_\mu$, then $(i, j) \leq (k, l)$ if $i \leq k$ and $j \leq l$.

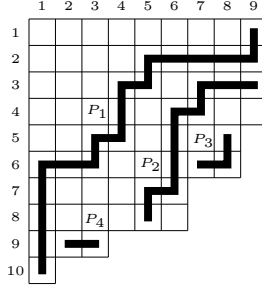
A **path on D_μ** is a path of contiguous boxes of the Young diagram D_μ which

- (i) moves only up or to the right, and
- (ii) begins on the lowest box of a column and ends on the rightmost box of a row.

Note that a path on D_μ may consist of only one box. In this case, the box must be a lower right corner of D_μ . We define the **greatest lower bound** of a path P , or $\text{glb}(P)$, to be the greatest lower bound of all the boxes of P . Explicitly, if the left endpoint of P is the box (i, j) and the right endpoint is the box (k, l) , then $\text{glb}(P)$ is the box $(k, j) \in D_\mu$. In particular, the endpoints of P determine $\text{glb}(P)$. We impose the following order on paths on D_μ : if P, P' are paths on D_μ , then $P \leq P'$ if $\text{glb}(P) \leq \text{glb}(P')$.

Denote by \mathcal{F}_μ the set of families of nonintersecting paths on D_μ . Let $F \in \mathcal{F}_\mu$. Define the **support** of F , $\text{Supp}(F)$, to be the set of all boxes in all paths of F .

Example 5.1. A family $F = \{P_1, P_2, P_3, P_4\}$ of nonintersecting paths on D_μ , $\mu = (9, 9, 9, 9, 8, 8, 6, 6, 3, 1)$. We have $\text{glb}(P_1) = (1, 1)$, $\text{glb}(P_2) = (3, 5)$, $\text{glb}(P_3) = (5, 7)$, $\text{glb}(P_4) = (9, 2)$, and $P_1 \leq P_2 \leq P_3$, $P_1 \leq P_4$.



Lemma 5.2. $\text{Supp}(F)$ uniquely determines the paths of F .

Proof. In other words, if $\text{Supp}(F') = \text{Supp}(F)$, then $F' = F$. Our proof is by decreasing induction on the number m of paths of F . If P is a minimal path of F , then one sees that P must also be a path of F' . In particular, the result for $m = 1$ holds. If $m > 1$, then $\text{Supp}(F' \setminus \{P\}) = \text{Supp}(F') \setminus \text{Supp}(\{P\}) = \text{Supp}(F) \setminus \text{Supp}(\{P\}) = \text{Supp}(F \setminus \{P\})$, and $F \setminus \{P\}$ has $m - 1$ paths. Thus by induction $F' \setminus \{P\} = F \setminus \{P\}$, which completes the proof. \square

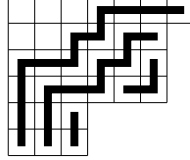
Let $\lambda \leq \mu$, i.e., $\lambda_i \leq \mu_i$ for all i . Then D_λ naturally embeds in D_μ in such a way that both Young diagrams share the same top left corner. In this way, D_λ can be viewed as a subset of D_μ .

Lemma 5.3. There exists $F \in \mathcal{F}_\mu$ such that $\text{Supp}(F) = D_\mu \setminus D_\lambda$.

Proof. We use decreasing induction on the number of boxes of $D_\mu \setminus D_\lambda$. One can form a path P on D_μ consisting of boxes $(i, j) \in D_\mu \setminus D_\lambda$ such that $(i - 1, j - 1) \notin D_\mu \setminus D_\lambda$. This path moves along an upper left boundary of $D_\mu \setminus D_\lambda$. Now $D' = D_\lambda \cup P$ is a Young diagram contained in D_μ , and $D_\mu \setminus D'$ has fewer boxes than $D_\mu \setminus D_\lambda$. Thus by induction there exists $F' \in \mathcal{F}_\mu$ such that $\text{Supp}(F') = D_\mu \setminus D' = (D_\mu \setminus D_\lambda) \setminus P$. Set $F = F' \cup \{P\}$. \square

By Lemma 5.2, the family F of Lemma 5.3 is uniquely determined. We call this family the **top family on D_μ for λ** and denote it by $F_{\lambda,\mu}^{\text{top}}$.

Example 5.4. The family $F_{\lambda,\mu}^{\text{top}}$, where $\mu = (7, 6, 6, 6, 3, 3)$, $\lambda = (3, 2)$.



If $F \in F_\mu$, then we define $\text{twist}(F) = \{\text{glb}(P) \mid P \text{ is a path of } F\}$, and we call this the **twist** of F . (In Example 5.1, $\text{twist}(F) = \{(1, 1), (3, 5), (5, 7), (9, 2)\}$.)

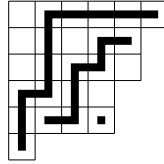
Lemma 5.5. If $\text{twist}(F_{\nu,\mu}^{\text{top}}) = \text{twist}(F_{\lambda,\mu}^{\text{top}})$, then $\nu = \lambda$.

Proof. By decreasing induction on μ . Let $b \in D_\mu$ be a maximal element of $\text{twist}(F_{\lambda,\mu}^{\text{top}})$, and thus also of $\text{twist}(F_{\nu,\mu}^{\text{top}})$. Then $b = \text{glb}(P)$ for some maximal path P of $F_{\lambda,\mu}^{\text{top}}$, which runs along the lower-right boundary of D_μ . Likewise, $b = \text{glb}(P')$ for some maximal path P' of $F_{\nu,\mu}^{\text{top}}$, which runs along the lower-right boundary of D_μ . Since P' and P are paths with the same greatest lower bound, and they both run along the lower right boundary of D_μ , $P' = P$.

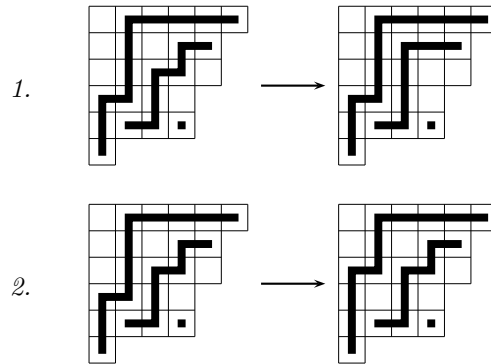
Now $D_\mu \setminus P = D_{\mu'}$, for some $\mu' < \mu$. We have that $F_{\lambda,\mu'}^{\text{top}} = F_{\lambda,\mu}^{\text{top}} \setminus \{P\}$, $F_{\nu,\mu'}^{\text{top}} = F_{\lambda,\mu}^{\text{top}} \setminus \{P\}$, and thus $\text{twist}(F_{\nu,\mu'}) = \text{twist}(F_{\nu,\mu}) \setminus \{b\} = \text{twist}(F_{\lambda,\mu}) \setminus \{b\} = \text{twist}(F_{\lambda,\mu'})$. Therefore $\nu = \lambda$ follows by induction. \square

Suppose that D_μ contains the four boxes (i, j) , $(i + 1, j)$, $(i, j + 1)$, and $(i + 1, j + 1)$ (which make up a square), and that F contains $(i + 1, j)$, $(i, j + 1)$, and $(i + 1, j + 1)$, but not (i, j) . One checks that these three boxes must lie on the same path P of F . We apply a **ladder move** to F by altering P as follows: the box $(i + 1, j + 1)$ of P is removed and replaced with the box (i, j) , thus obtaining a new path P' on D_μ . The resulting path P' , combined with the paths of F other than P , form a new family of nonintersecting paths F' on D_μ . We denote this ladder move by $F \rightarrow F'$. Note that a ladder move is invertible. We call the inverse of a ladder move a **reverse ladder move**.

Example 5.6. Let F be the following family of nonintersecting paths on D_μ , $\mu = (6, 5, 5, 4, 4, 1)$:

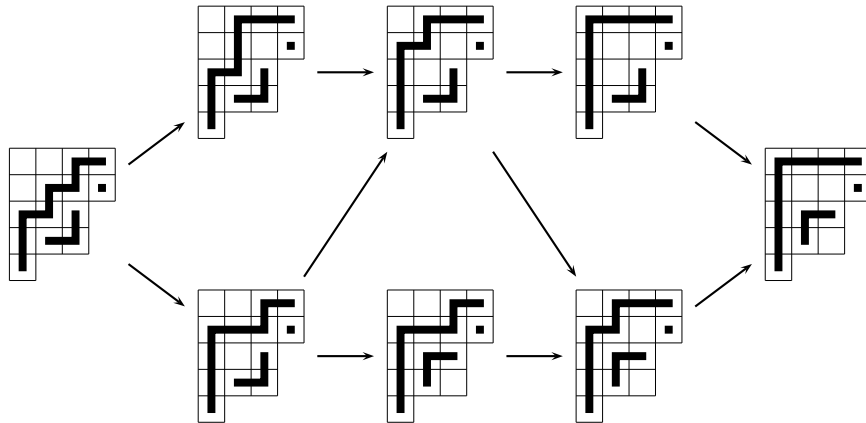


The following two ladder moves can be applied to F :



A family of nonintersecting paths on D_μ for λ is an element of \mathcal{F}_μ which can be obtained by applying a succession of ladder moves to $F_{\lambda, \mu}^{\text{top}}$. The set of all families of nonintersecting paths on D_μ for λ is denoted by $\mathcal{F}_{\lambda, \mu}$.

Example 5.7. The following diagram shows $\mathcal{F}_{\lambda, \mu}$, as well as all possible ladder moves, where $\mu = (4, 4, 3, 3, 1)$, $\lambda = (2, 1)$.

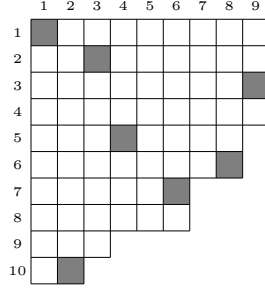


The Set $\mathcal{F}_{\lambda,\mu}''$ of Path Families

We say that a set of boxes $S \subset D_\mu$ is a **twisted chain** if for any two boxes p, p' in S :

1. p and p' lie on different rows and different columns of D_μ .
2. Either $p < p'$, $p' < p$, or $\{p, p'\}$ has no upper bound in D_μ .

Example 5.8. *The shaded boxes form a twisted chain in D_μ .*



Note that, for example, $\{(10, 2), (5, 4)\}$ has least upper bound $(10, 4)$, which is not in D_μ .

It follows from the definitions that if $F \in \mathcal{F}_\mu$, then $\text{twist}(F)$ is a twisted chain of D_μ . Define

$$\mathcal{F}'_{\lambda,\mu} = \{F \in \mathcal{F}_\mu \mid \text{twist}(F) = \text{twist}(F_{\lambda,\mu}^{\text{top}})\}.$$

Ultimately we will be interested in the set $\mathcal{F}_{\lambda,\mu}''$ of families of nonintersecting paths on D_μ , which is defined below. We introduce $\mathcal{F}'_{\lambda,\mu}$ in order to break up the proof that $\mathcal{F}_{\lambda,\mu} = \mathcal{F}_{\lambda,\mu}''$ into two parts: $\mathcal{F}_{\lambda,\mu} = \mathcal{F}'_{\lambda,\mu}$ and $\mathcal{F}'_{\lambda,\mu} = \mathcal{F}_{\lambda,\mu}''$.

Lemma 5.9. $\mathcal{F}'_{\lambda,\mu} = \mathcal{F}_{\lambda,\mu}$.

Proof. We remark that applying a ladder move to any family in $\mathcal{F}_{\lambda,\mu}$ does not alter its twist. Thus ladder moves preserve $\mathcal{F}'_{\lambda,\mu}$. Also $F_{\lambda,\mu}^{\text{top}} \in \mathcal{F}'_{\lambda,\mu}$. Since $\mathcal{F}_{\lambda,\mu}$ is by definition the smallest subset of \mathcal{F}_μ preserved by ladder moves which contains $F_{\lambda,\mu}^{\text{top}}$, $\mathcal{F}_{\lambda,\mu} \subset \mathcal{F}'_{\lambda,\mu}$.

Next suppose that $F \in \mathcal{F}'_{\lambda,\mu}$. By applying a succession of reverse ladder moves to F , $F_{\nu,\mu}^{\text{top}}$ can be obtained, for some $\nu \leq \mu$. Since reverse ladder moves do not alter twist, $\text{twist}(F_{\nu,\mu}^{\text{top}}) = \text{twist}(F) = \text{twist}(F_{\lambda,\mu}^{\text{top}})$. By Lemma 5.5, $\nu = \lambda$. Thus F can be obtained from $F_{\lambda,\mu}^{\text{top}}$ by applying ladder moves, which implies $F \in \mathcal{F}_{\lambda,\mu}$. Hence $\mathcal{F}'_{\lambda,\mu} \subset \mathcal{F}_{\lambda,\mu}$. \square

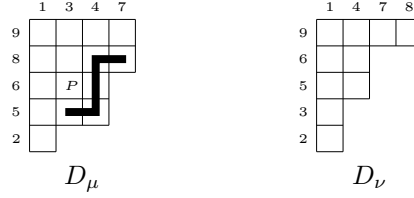
Let $\lambda \leq \mu$, and define $\alpha = \pi^{-1}(\lambda)$, $\beta = \pi^{-1}(\mu)$. In [Kre05], it is shown that there exists a unique twisted chain $S_{\lambda,\mu} = \{(x_1, y_1), \dots, (x_t, y_t)\} \subset D_\mu$ such that $\alpha = \beta \setminus \{\beta(d+1-x_1), \dots, \beta(d+1-x_t)\} \cup \{\beta'(y_1), \dots, \beta'(y_t)\}$. Define

$$\mathcal{F}_{\lambda,\mu}'' = \{F \in \mathcal{F}_\mu \mid \text{twist}(F) = S_{\lambda,\mu}\}.$$

It follows from the definitions that $\mathcal{F}_{\lambda,\mu}'' = \mathcal{F}_{\lambda,\mu}'$ if and only if $\text{twist}(F_{\lambda,\mu}^{\text{top}}) = S_{\lambda,\mu}$. We first prove this for the case where λ and μ are such that $D_\mu \setminus \bar{D}_\lambda$ consists of a single path.

Lemma 5.10. *Let P be a path which moves along the lower right border of D_μ , so that $D_\mu \setminus \{P\} = D_\nu$, for some $\nu < \mu$. Then $\text{twist}(F_{\nu,\mu}^{\text{top}}) = S_{\nu,\mu}$.*

Example 5.11. *Let $d = 5$, $n = 9$. Let $\mu = (4, 4, 3, 3, 1)$, and let P be the path on D_μ shown below. Then $D_\mu \setminus P = D_\nu$, where $\nu = (4, 2, 2, 1, 1)$. We have $F_{\nu,\mu}^{\text{top}} = \{P\}$, and $\text{twist}(F_{\nu,\mu}^{\text{top}}) = \text{glb}(P) = (2, 2)$. Let $\beta = \pi^{-1}(\mu) = \{2, 5, 6, 8, 9\}$, $\alpha = \pi^{-1}(\nu) = \{2, 3, 5, 6, 9\}$. Then $\alpha = \beta \setminus 8 \cup 3 = \beta \setminus \beta(5 + 1 - 2) \cup \beta'(2)$, so $S_{\nu,\mu} = (2, 2)$, which agrees with $\text{twist}(F_{\nu,\mu}^{\text{top}})$.*



Proof of Lemma 5.10. For $i = 1, \dots, d$, define $\bar{i} = d + 1 - i$; thus if a box is on row i of D_μ counting from the top, then it is on row \bar{i} counting from the bottom. Let $(x, y) = \text{glb}(P) = \text{twist}(F_{\nu,\mu}^{\text{top}})$. Let $\beta = \pi^{-1}(\mu)$ and $\alpha = \beta \setminus \beta(\bar{x}) \cup \beta'(y)$. By definition, $S_{\pi(\alpha),\mu} = (x, y)$. We shall obtain the result by showing that $\pi(\alpha) = \nu$.

Note that

$$D_\mu = \{(u, v) \in \{1, \dots, d\} \times \{1, \dots, n - d\} \mid \beta(\bar{u}) > \beta'(v)\}.$$

Let $w = \min\{u \mid \beta(u) > \beta'(y)\}$, and note that $w \leq \bar{x}$. We have

$$\alpha = \{\beta(1), \dots, \beta(w - 1), \beta'(y), \beta(w), \dots, \beta(\bar{x} - 1), \widehat{\beta(\bar{x})}, \beta(\bar{x} + 1), \dots, \beta(d)\}.$$

Therefore,

$$\alpha(i) = \begin{cases} \beta(i), & i < w \text{ or } i > \bar{x} \\ \beta(i - 1), & w < i \leq \bar{x} \\ \beta'(y), & i = w. \end{cases}$$

We have

$$\begin{aligned} (\pi(\alpha))_i &= \alpha(\bar{i}) - (\bar{i}) \\ &= \begin{cases} \beta(\bar{i}) - \bar{i}, & \bar{i} < w \text{ or } \bar{i} > \bar{x} \\ \beta(\bar{i} - 1) - \bar{i}, & w < \bar{i} \leq \bar{x} \\ \beta'(y) - \bar{w}, & \bar{i} = w \end{cases} \\ &= \begin{cases} \mu_i, & \bar{i} < w \text{ or } \bar{i} > \bar{x} \\ \mu_{i+1} - 1, & w < \bar{i} \leq \bar{x} - 1 \\ y - 1, & \bar{i} = w \end{cases} \\ &= \nu_i, \end{aligned}$$

where we use the facts that $\beta(\bar{i}) - (\bar{i}) = \mu_i$, $\beta(\bar{i}-1) - (\bar{i}) = \beta(\overline{i+1}) - (\overline{i+1}) - 1 = \mu_{i+1} - 1$, and $\beta'(y) - y = \bar{w} - 1$. \square

Lemma 5.12. *For any $\lambda \leq \mu$, $\text{twist}(F_{\lambda,\mu}^{\text{top}}) = S_{\lambda,\mu}$.*

Proof. Define $\beta = \pi^{-1}(\mu)$, $\alpha = \pi^{-1}(\lambda)$, and let $\{(x_1, y_1), \dots, (x_t, y_t)\} = \text{twist}(F_{\lambda,\mu}^{\text{top}})$. We wish to show that $\alpha = \beta \setminus \{\beta(d+1-x_1), \dots, \beta(d+1-x_t)\} \cup \{\beta'(y_1), \dots, \beta'(y_t)\}$.

We use decreasing induction on t , the number of paths in $F_{\lambda,\mu}^{\text{top}}$. For $t = 1$, the result is identically Lemma 5.10. Suppose that $t > 1$. Let P be a maximal path of $F_{\lambda,\mu}^{\text{top}}$. Then P runs along the lower right border of D_μ , so $D_\mu \setminus P = D_\nu$, for some $\nu < \mu$. Define $\gamma = \pi^{-1}(\nu)$, and assume that $(x_t, y_t) = \text{glb}(P)$ (re-indexing if necessary). By Lemma 5.10,

$$\gamma = \beta \setminus \beta(d+1-x_t) \cup \beta'(y_t). \quad (10)$$

Now $F_{\lambda,\nu}^{\text{top}} = F_{\lambda,\mu}^{\text{top}} \setminus \{P\}$, whose twist is equal to $\{(x_1, y_1), \dots, (x_{t-1}, y_{t-1})\}$. Thus by induction,

$$\alpha = \gamma \setminus \{\beta(d+1-x_1), \dots, \beta(d+1-x_{t-1})\} \cup \{\beta'(y_1), \dots, \beta'(y_{t-1})\}. \quad (11)$$

Combining (10) and (11), we arrive at the result. \square

From Lemma 5.12, we obtain

Lemma 5.13. $\mathcal{F}_{\lambda,\mu}'' = \mathcal{F}'_{\lambda,\mu}$.

Now Lemmas 5.9 and 5.13 imply

Lemma 5.14. $\mathcal{F}_{\lambda,\mu}'' = \mathcal{F}_{\lambda,\mu}$.

6 From Path Families to Semistandard Young Tableaux

Lemma 4.2 follows from a result of [KR03], [KL03], and [Kre05] which gives an equivariant Gröbner degeneration of a Schubert variety in the neighborhood of a T -fixed point to a union of coordinate subspaces. However, whereas the result of [KR03], [KL03], and [Kre05] is expressed in terms of the set $\mathcal{F}_{\lambda,\mu}''$ of families of nonintersecting paths on Young diagrams, our results, and in particular Lemma 4.2, require the semistandard Young tableaux $\text{SSYT}_{\lambda,\mu}$. Most of the previous section and this one are taken up in showing the equivalence between these two combinatorial models. In the previous section we showed that $\mathcal{F}_{\lambda,\mu}'' = \mathcal{F}_{\lambda,\mu}$. In this section, we introduce a new model, ‘subsets of Young diagrams’. We will be interested in a certain set of subsets of the Young diagram D_μ , which we denote by $\mathcal{D}_{\lambda,\mu}$. We show that $\mathcal{F}_{\lambda,\mu}$ and $\text{SSYT}_{\lambda,\mu}$ are both equivalent to $\mathcal{D}_{\lambda,\mu}$, and thus are equivalent to each other. We summarize the steps we take in showing the equivalence between $\mathcal{F}_{\lambda,\mu}''$ and $\text{SSYT}_{\lambda,\mu}$ as follows:

$$\mathcal{F}_{\lambda,\mu}'' = \mathcal{F}_{\lambda,\mu} \longleftrightarrow \mathcal{D}_{\lambda,\mu} \longleftrightarrow \text{SSYT}_{\lambda,\mu}$$

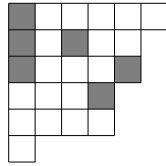
The subsets $\mathcal{D}_{\lambda,\mu}$, which were discovered independently by Ikeda-Naruse [IN], are similar to RC graphs or reduced pipe dreams [BB93, FK94, KM05] for Grassmannian permutations.

Subsets of Young Diagrams

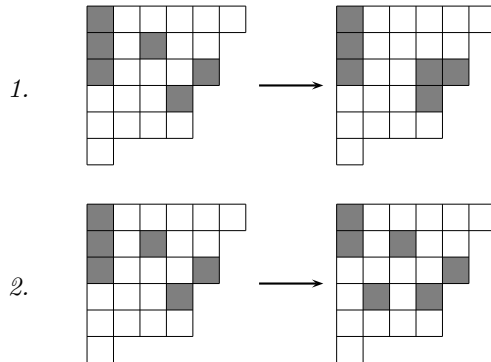
Let μ be a partition and D_μ the corresponding Young diagram. Lemma 5.2 tells us that a family of nonintersecting paths on D_μ is completely characterized by its support. This suggests that in order to study families of nonintersecting paths on D_μ , it suffices to study the supports of these path families (or the complements in D_μ of their supports). This motivates the following definitions.

A **subset of D_μ** is a set of boxes in D_μ . Let D be a subset of D_μ . Suppose that D_μ contains the four boxes (i, j) , $(i + 1, j)$, $(i, j + 1)$, and $(i + 1, j + 1)$ (which make up a square), but of these four boxes, D only contains (i, j) . Then a **ladder move** removes (i, j) from D and replaces it with $(i + 1, j + 1)$, thereby obtaining a new subset D' of D_μ . We denote this ladder move by $D \rightarrow D'$. Note that a ladder move is invertible. We call the inverse of a ladder move a **reverse ladder move**.

Example 6.1. Let D be the following subset of D_μ , $\mu = (6, 5, 5, 4, 4, 1)$ (boxes of D are shaded):

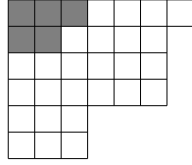


The following two ladder moves can be applied to F :



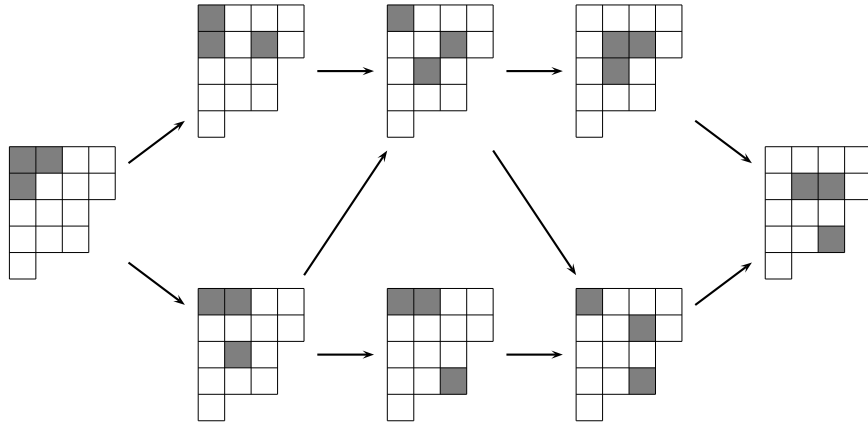
Let λ be a partition with $\lambda \leq \mu$, i.e., $\lambda_i \leq \mu_i$ for all i . Embed the Young diagram D_λ in D_μ in such a way that both D_λ and D_μ share the same top left corners. The subset of D_μ consisting of all boxes in this embedded Young diagram is called the **top subset of D_μ for λ** and denoted by $D_{\lambda,\mu}^{\text{top}}$.

Example 6.2. The subset $D_{\lambda,\mu}^{\text{top}}$, where $\mu = (7, 6, 6, 6, 3, 3)$, $\lambda = (3, 2)$.



A **subset of D_μ for λ** is a subset of D_μ which can be obtained by applying a succession of ladder moves to the top subset of D_μ for λ . The set of all subsets of D_μ for λ is denoted by $\mathcal{D}_{\lambda,\mu}$.

Example 6.3. The following diagram shows $\mathcal{D}_{\lambda,\mu}$, as well as all possible ladder moves, where $\mu = (4, 4, 3, 3, 1)$, $\lambda = (2, 1)$.

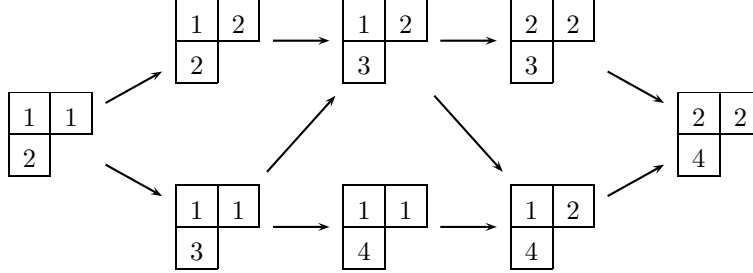


Semistandard Young Tableaux

A **ladder move** on a semistandard Young tableau is an operation which increments one of the entries of the tableau by 1 and results in a semistandard Young tableau. Note that a ladder move is invertible. We call the inverse of a ladder move a **reverse ladder move**. Recall that a semistandard Young tableau is **on μ** if each of its entries satisfies (1). Let $\lambda \leq \mu$. Define the **top semistandard Young tableau on μ of shape λ** to be the (semistandard) Young tableau of shape λ whose i -th row is filled with i 's. This definition does not depend on μ .

The set of all semistandard Young tableaux on μ of shape λ , $\text{SSYT}_{\lambda,\mu}$, is precisely the set of semistandard Young tableaux on μ which can be obtained by applying sequences of ladder moves to the top semistandard Young tableaux on μ of shape λ . This follows from the facts that (i) by applying a sequence of reverse ladder moves to any semistandard Young tableau of shape λ , the top semistandard Young tableaux on μ of shape λ can be obtained, and (ii) reverse ladder moves preserve (1).

Example 6.4. The following diagram shows $\text{SSYT}_{\lambda,\mu}$, as well as all possible ladder moves, where $\mu = (4, 4, 3, 3, 1)$, $\lambda = (2, 1)$.



The Equivalences $\mathcal{F}_{\lambda,\mu} \longleftrightarrow \mathcal{D}_{\lambda,\mu}$ and $\mathcal{D}_{\lambda,\mu} \longleftrightarrow \text{SSYT}_{\lambda,\mu}$

Recall that

\mathcal{F}_μ = the set of families of nonintersecting paths on D_μ
 $\mathcal{F}_{\lambda,\mu}$ = the set of families of nonintersecting paths on D_μ for λ
 $F_{\lambda,\mu}^{\text{top}}$ = the top family of nonintersecting paths on D_μ for λ

\mathcal{D}_μ = the set of subsets of D_μ
 $\mathcal{D}_{\lambda,\mu}$ = the set of subsets of D_μ for λ
 $D_{\lambda,\mu}^{\text{top}}$ = the top subset of D_μ for λ

$\text{SSYT}_{\lambda,\mu}$ = the set of semistandard Young tableaux on μ of shape λ

$P_{\lambda,\mu}^{\text{top}}$ = the top semistandard Young tableaux on μ of shape λ

Define $h : \mathcal{F}_{\lambda,\mu} \rightarrow \mathcal{D}_{\lambda,\mu}$ by $h(F) = D_\mu \setminus \text{Supp}(F)$. Then h maps $F_{\lambda,\mu}^{\text{top}}$ to $D_{\lambda,\mu}^{\text{top}}$ and commutes with ladder moves. Therefore it restricts to a map from $\mathcal{F}_{\lambda,\mu}$ to $\mathcal{D}_{\lambda,\mu}$. Injectivity of h follows immediately from Lemma 5.2. To show surjectivity, assume that $h(F) = D$, for some $D \in \mathcal{D}_\mu$, $F \in \mathcal{F}_\mu$. Suppose that a ladder move is applied to D to obtain D' . Then there is a corresponding ladder move which when applied to F yields F' such that $h(F') = D'$. Thus surjectivity of $h : \mathcal{F}_{\lambda,\mu} \rightarrow \mathcal{D}_{\lambda,\mu}$ follows by induction on number of ladder moves.

We next give a bijection g from $\mathcal{D}_{\lambda,\mu}$ to $\text{SSYT}_{\lambda,\mu}$. Let $D \in \mathcal{D}_{\lambda,\mu}$. By applying a sequence of reverse ladder moves to D , $D_{\lambda,\mu}^{\text{top}}$ can be obtained. This sequence of reverse ladder moves takes the box (x, y) of D to some box $(i_{x,y}, j_{x,y})$ of $D_{\lambda,\mu}^{\text{top}}$. Note that $(i_{x,y}, j_{x,y})$ depends only on x and y (and D), and not on the sequence of reverse ladder moves. Let $g(D)$ be the tableau of shape λ which, for each box (x, y) of D , contains entry x in box $(i_{x,y}, j_{x,y})$. The definitions of ladder and reverse ladder moves on subsets of D_μ imply that $g(D)$ is semistandard. Also,

since $(x, y) \in D_\mu$, $y \leq \mu(x)$; thus $y - x = j_{x,y} - i_{x,y}$ implies $x + j_{x,y} - i_{x,y} \leq \mu(x)$. Therefore $g(D)$ is on μ .

To show that g is bijective, we give the inverse map f from $\text{SSYT}_{\lambda,\mu}$ to $\mathcal{D}_{\lambda,\mu}$. Let $P \in \text{SSYT}_{\lambda,\mu}$, and let $f(P)$ be the subset of D_μ which, for each entry $x \in P$, contains box $(x, x - r(x) + c(x))$ of D_μ (where recall that $(r(x), c(x))$ is the box of x in P). Since P is on μ , $(x, x - r(x) + c(x)) \in D_\mu$; thus $f(P)$ is indeed in \mathcal{D}_μ . Let $P_{\lambda,\mu}^{\text{top}}$ denote the top semistandard Young tableaux on μ of shape λ . Because $f(P_{\lambda,\mu}^{\text{top}}) = D_{\lambda,\mu}^{\text{top}}$ and f commutes with ladder moves, $f(P) \in \mathcal{D}_{\lambda,\mu}$.

Proof of Lemma 4.2

Let $\lambda = \pi(\alpha)$, $\mu = \pi(\beta)$. Let $\{v_{a,b} \mid (a, b) \in \beta' \times \beta\} \subset \mathcal{O}_\beta$ denote the basis dual to the basis of linear forms $\{y_{a,b} \mid (a, b) \in \beta' \times \beta\} \subset \mathcal{O}_\beta^*$. For $F \in \mathcal{F}_\mu$, define

$$W_F = \text{Span}(\{v_{\beta'(z), \beta(d+1-x)} \mid (x, z) \in \text{Supp}(F)\} \cup \{v_{a,b} \mid (a, b) \in \beta' \times \beta, a > b\}).$$

In [KR03, KL03, Kre05], an explicit equivariant bijection (which is called the bounded RSK in [Kre05]) is constructed from $\mathbb{C}[\bigcup_{F \in \mathcal{F}_{\lambda,\mu}} W_F]$ to $\mathbb{C}[Y_{\alpha,\beta}]$. Thus in light of Lemma 5.14,

$$\text{Char}(\mathbb{C}[Y_{\alpha,\beta}]) = \text{Char} \left(\mathbb{C} \left[\bigcup_{F \in \mathcal{F}_{\lambda,\mu}} W_F \right] \right) = \text{Char} \left(\mathbb{C} \left[\bigcup_{F \in \mathcal{F}_{\lambda,\mu}} W_F \right] \right). \quad (12)$$

For $x \in \{1, \dots, d\}$, $z \in \{1, \dots, n - d\}$, we have that $(x, z) \in D_\mu \iff z \leq \mu_x \iff \beta'(z) < \beta(d+1-x)$ (see Remark 2.4). Thus $\{(a, b) \in \beta' \times \beta \mid a < b\}$ can be expressed as $\{(\beta'(z), \beta(d+1-x)) \mid (x, z) \in D_\mu\}$. Let $F \in \mathcal{F}_{\lambda,\mu}$, $D = h(F)$, and $P = g(D)$. Since $\text{Supp}(F)$ and D are complements in D_μ ,

$$\begin{aligned} \beta' \times \beta &= \{(\beta'(z), \beta(d+1-x)) \mid (x, z) \in \text{Supp}(F)\} \dot{\cup} \{(a, b) \in \beta' \times \beta \mid a > b\} \\ &\quad \dot{\cup} \{(\beta'(z), \beta(d+1-x)) \mid (x, z) \in D\}. \end{aligned}$$

Therefore

$$\begin{aligned} W_F &= V(\{y_{\beta'(z), \beta(d+1-x)} \mid (x, z) \in D\}) \\ &= V(\{y_{\beta'(x-r(x)+c(x), \beta(d+1-x))} \mid x \in P\}) \\ &= W_P. \end{aligned}$$

Consequently, $\bigcup_{F \in \mathcal{F}_{\lambda,\mu}} W_F = \bigcup_{P \in \text{SSYT}_{\lambda,\mu}} W_P = W_{\alpha,\beta}$, and thus

$$\text{Char} \left(\mathbb{C} \left[\bigcup_{F \in \mathcal{F}_{\lambda,\mu}} W_F \right] \right) = \text{Char} \left(\mathbb{C} \left[\bigcup_{P \in \text{SSYT}_{\lambda,\mu}} W_P \right] \right) = \text{Char}(W_{\alpha,\beta}). \quad (13)$$

Combining (12) and (13), we obtain $\text{Char}(\mathbb{C}[Y_{\alpha,\beta}]) = \text{Char}(\mathbb{C}[W_{\alpha,\beta}])$. By (2), $[Y_{\alpha,\beta}]_{\mathbb{K}} = [W_{\alpha,\beta}]_{\mathbb{K}}$.

7 Computing N_S

In this section we prove Lemma 4.4.

For a set-valued tableau S , we denote by $S_{i,j}$ the set of entries of S which are contained in the box with row and column numbers i and j respectively, and we denote by $S_{i,j,1}, \dots, S_{i,j,r}$ the entries of $S_{i,j}$, which we assume are listed in increasing order. We define $\mathcal{M}_k(S)$ to be the k -element subsets of $\text{SSYT}(S)$, and $\mathcal{N}_k(S)$ the k -element subsets of $\text{SSYT}(S)$ whose unions equals S . By definition, $N_{S,k} = |\mathcal{N}_k(S)|$.

Define a **generalized set-valued tableau** to be the assignment of a *possibly empty* set of positive integers to each box of a Young diagram. If S and R are two set-valued tableaux of the same shape, then the difference $S \setminus R$ is defined to be the generalized set-valued tableau of the same shape as S (or R) with $(S \setminus R)_{i,j} = S_{i,j} \setminus R_{i,j}$ for all i, j . Recall that we write $R \subset S$ to indicate that $R_{i,j} \subset S_{i,j}$ for all i, j . If $R \subset S$, and S is clear from the context, then we also denote $S \setminus R$ by \overline{R} . The only generalized set-valued tableaux which appear in this section in which boxes may be empty occur explicitly as differences of set-valued tableaux.

Lemma 7.1.
$$N_S = \sum_{R \subset S} (-1)^{|\overline{R}|} - \sum_{\substack{R \subset S \\ \text{SSYT}(R) = \emptyset}} (-1)^{|\overline{R}|}.$$

Proof. If $q_S = 0$ (i.e., $\text{SSYT}(S) = \emptyset$), then for every set-valued tableau $R \subset S$, $\text{SSYT}(R) = \emptyset$; thus the result is trivially true. Assume $q_S \neq 0$.

For $k \geq 1$, we have that $\mathcal{N}_k(S) = \mathcal{M}_k(S) \setminus \bigcup_{\substack{R \subset S \\ R \neq S}} \mathcal{M}_k(R)$; by the inclusion-exclusion principle,

$$|\mathcal{N}_k(S)| = \sum_{R \subset S} -1^{|\overline{R}|} \cdot |\mathcal{M}_k(R)| = \sum_{R \subset S} (-1)^{|\overline{R}|} \binom{|\text{SSYT}(R)|}{k} = \sum_{R \subset S} (-1)^{|\overline{R}|} \binom{q_R}{k}$$

where we use the convention $\binom{a}{b} = 0$ if $a < b$. Thus,

$$\begin{aligned} N_S &= \sum_{k=1}^{q_S} (-1)^{k+1} |\mathcal{N}_k(S)| \\ &= \sum_{k=1}^{q_S} (-1)^{k+1} \sum_{R \subset S} (-1)^{|\overline{R}|} \binom{q_R}{k} \\ &= \sum_{R \subset S} (-1)^{|\overline{R}|} \sum_{k=1}^{q_S} (-1)^{k+1} \binom{q_R}{k}. \end{aligned}$$

The result now follows from the fact that for any $R \subset S$, $q_R \leq q_S$, and therefore

$$\sum_{k=1}^{q_S} (-1)^{k+1} \binom{q_R}{k} = \begin{cases} 1 & \text{if } q_R \neq 0 \\ 0 & \text{if } q_R = 0. \end{cases}$$

□

The following Lemma gives the value of the first summation in Lemma 7.1.

Lemma 7.2. $\sum_{R \subset S} (-1)^{\overline{R}} = (-1)^{|S| + \|S\|}$.

Proof.

$$\begin{aligned}
\sum_{R \subset S} (-1)^{\overline{R}} &= \sum_{\substack{R_{1,1} \subset S_{1,1} \\ R_{1,1} \neq \emptyset}} \cdots \sum_{\substack{R_{u,v} \subset S_{u,v} \\ R_{u,v} \neq \emptyset}} (-1)^{|S_{1,1} \setminus R_{1,1}| + \cdots + |S_{u,v} \setminus R_{u,v}|} \\
&= \prod_{i,j} \sum_{\substack{R_{i,j} \subset S_{i,j} \\ R_{i,j} \neq \emptyset}} (-1)^{|S_{i,j} \setminus R_{i,j}|} \\
&= \prod_{i,j} \sum_{R'_{i,j} \subsetneq S_{i,j}} (-1)^{|R'_{i,j}|} \\
&= \prod_{i,j} \sum_{k=0}^{|S_{i,j}|-1} (-1)^k \binom{|S_{i,j}|}{k} \\
&= \prod_{i,j} (-1)^{|S_{i,j}|-1} \\
&= (-1)^{|S| + \|S\|}.
\end{aligned}$$

□

If S is semistandard, then for every $R \subset S$, $\text{SSYT}(R) \neq \emptyset$. Thus Lemmas 7.1 and 7.2 imply Lemma 4.4(i).

The following Lemma gives the value of the second summation in Lemma 7.1 when S is not semistandard. Lemmas 7.1, 7.2, and the following Lemma imply Lemma 4.4(ii).

Lemma 7.3. *If S is not semistandard, then* $\sum_{\substack{R \subset S \\ \text{SSYT}(R) = \emptyset}} (-1)^{|\overline{R}|} = (-1)^{|S| + \|S\|}$.

Proof. Define $\mathcal{Z}(S) = \{R \subset S \mid \text{SSYT}(R) = \emptyset\}$, so that

$$\sum_{\substack{R \subset S \\ \text{SSYT}(R) = \emptyset}} (-1)^{|\overline{R}|} = \sum_{R \in \mathcal{Z}(S)} (-1)^{|\overline{R}|}. \quad (14)$$

We make a series of reductions.

Let x and y be the row and column numbers of a box of S where semistandardness is violated either on the top or left, but not on the right or bottom. Define

$$\begin{aligned}
\mathcal{Z}'(S) &= \{R \in \mathcal{Z}(S) \mid \{S_{x,y,1}\} = R_{x,y}\} \\
\mathcal{Z}''(S) &= \{R \in \mathcal{Z}(S) \mid \{S_{x,y,1}\} \subsetneq R_{x,y}\} \\
\mathcal{Z}'''(S) &= \{R \in \mathcal{Z}(S) \mid S_{x,y,1} \notin R_{x,y}\}.
\end{aligned}$$

Then $\mathcal{Z}(S) = \mathcal{Z}'(S) \dot{\cup} \mathcal{Z}''(S) \dot{\cup} \mathcal{Z}'''(S)$. Consider the bijection from $\mathcal{Z}''(S)$ to $\mathcal{Z}'''(S)$ defined by $R \mapsto R \setminus S_{x,y,1}$. The set-valued tableaux paired under this bijection contribute opposite signs to (14). Thus

$$\sum_{R \in \mathcal{Z}(S)} (-1)^{|\overline{R}|} = \sum_{R \in \mathcal{Z}'(S)} (-1)^{|\overline{R}|}. \quad (15)$$

Let $g = S_{x,y,1}$. Define

$$\mathcal{Y}'(S) = \{R \in \mathcal{Z}'(S) \mid R_{x-1,y,k} < g, R_{x,y-1,l} \leq g, \text{ some } k, l\}$$

$$\mathcal{Y}''(S) = \{R \in \mathcal{Z}'(S) \mid R_{x-1,y,k} \geq g, \text{ all } k\}$$

$$\mathcal{Y}'''(S) = \{R \in \mathcal{Z}'(S) \mid R_{x,y-1,k} > g, \text{ all } k\}.$$

Then $\mathcal{Z}'(S) = \mathcal{Y}'(S) \dot{\cup} (\mathcal{Y}''(S) \cup \mathcal{Y}'''(S))$. Therefore

$$\begin{aligned} \sum_{R \in \mathcal{Z}'(S)} (-1)^{|\overline{R}|} &= \sum_{R \in \mathcal{Y}'(S)} (-1)^{|\overline{R}|} + \sum_{R \in \mathcal{Y}''(S)} (-1)^{|\overline{R}|} \\ &\quad + \sum_{R \in \mathcal{Y}'''(S)} (-1)^{|\overline{R}|} - \sum_{R \in \mathcal{Y}''(S) \cap \mathcal{Y}'''(S)} (-1)^{|\overline{R}|}. \end{aligned} \quad (16)$$

We compute the last three summations on the right hand side of (16). Assume that $\mathcal{Y}''(S) \neq \emptyset$. Let S' be the tableaux obtained from S by removing all entries other than g from $S_{x,y}$ and all entries less than g from $S_{x-1,y}$. Then $\mathcal{Y}''(S) = \{R \subset S'\}$. Thus

$$\begin{aligned} \sum_{R \in \mathcal{Y}''(S)} (-1)^{|\overline{R}|} &= \sum_{R \subset S'} (-1)^{|S \setminus R|} \\ &= \sum_{R \subset S'} (-1)^{|S \setminus S'| + |S' \setminus R|} \\ &= (-1)^{|S \setminus S'|} \sum_{R \subset S'} (-1)^{|S' \setminus R|} \\ &= (-1)^{|S \setminus S'|} (-1)^{|S'| + \|S'\|} \\ &= (-1)^{|S| + \|S\|}, \end{aligned}$$

where the seconds to last equality follows from Lemma 7.2, and the last equality from the fact that $\|S'\| = \|S\|$. In a similar manner, one shows that that

$$\begin{aligned} \sum_{R \in \mathcal{Y}'''(S)} (-1)^{|\overline{R}|} &= (-1)^{|S| + \|S\|} \quad \text{if } \mathcal{Y}'''(S) \neq \emptyset, \text{ and} \\ \sum_{R \in \mathcal{Y}''(S) \cap \mathcal{Y}'''(S)} &= (-1)^{|S| + \|S\|} \quad \text{if } \mathcal{Y}''(S) \cap \mathcal{Y}'''(S) \neq \emptyset. \end{aligned}$$

Either $\mathcal{Y}''(S)$ or $\mathcal{Y}'''(S)$ must be nonempty, and if both of them are nonempty, then so must be their intersection. It follows that

$$\sum_{R \in \mathcal{Z}'(S)} (-1)^{|\overline{R}|} = (-1)^{|S| + \|S\|} + \sum_{R \in \mathcal{Y}'(S)} (-1)^{|\overline{R}|}. \quad (17)$$

Define

$$\mathcal{X}(S) = \{R \in \mathcal{Y}'(S) \mid R_{x-1,y,k} < g, R_{x,y-1,l} \leq g, \text{ all } k, l\}.$$

Let $A = \{a_1, \dots, a_r\}$ be the entries of $S_{x-1,y}$ which are greater than or equal to g , let $B = \{b_1, \dots, b_s\}$ be the entries of $S_{x,y-1}$ which are greater than g , and let $t = r + s$. For each $R \in \mathcal{X}(S)$, define $\mathcal{Y}'_R(S)$ to be all the set-valued tableaux obtained by adding elements of A to $R_{x-1,y}$ and elements of B to $S_{x,y-1}$. Then

$$\mathcal{Y}'(S) = \bigcup_{R \in \mathcal{X}(S)} \mathcal{Y}'_R(S),$$

and

$$\begin{aligned} \sum_{R \in \mathcal{Y}'(S)} (-1)^{|\bar{R}|} &= \sum_{R \in \mathcal{X}(S)} \sum_{Q \in \mathcal{Y}'_R(S)} (-1)^{|S \setminus Q|} \\ &= \sum_{R \in \mathcal{X}(S)} (-1)^{|S \setminus R|} \left((-1)^0 \binom{t}{0} + (-1)^1 \binom{t}{1} + \dots + (-1)^t \binom{t}{t} \right) \\ &= 0. \end{aligned} \tag{18}$$

Combining (14), (15), (17), and (18), we obtain the result. \square

References

- [AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a p th root of unity and of semisimple groups in characteristic p : independence of p* , *Astérisque* (1994), no. 220, 321.
- [BB93] Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, *Experiment. Math.* **2** (1993), no. 4, 257–269.
- [Ber92] Nantel Bergeron, *A combinatorial construction of the Schubert polynomials*, *J. Combin. Theory Ser. A* **60** (1992), no. 2, 168–182.
- [Bi99] Sara C. Billey, *Kostant polynomials and the cohomology ring for G/B* , *Duke Math. J.* **96** (1999), no. 1, 205–224.
- [BKTY05] Anders S. Buch, Andrew Kresch, Harry Tamvakis, and Alexander Yong, *Grothendieck polynomials and quiver formulas*, *Amer. J. Math.* **127** (2005), no. 3, 551–567.
- [Bu02] A. Buch, *A Littlewood-Richardson rule for the K -theory of Grassmannians*, *Acta Math.* **189** (2002), no. 1, 37–78.
- [FK94] Sergey Fomin and Anatol N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, *Proceedings of the Sixth Conference in Formal Power Series and Algebraic Combinatorics, DIMACS, 1994*, pp. 183–190.

- [FK96] ———, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. **153** (1996), no. 1-3, 123–143.
- [GR04a] S. Ghorpade and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in the symplectic Grassmannian*, preprint, arXiv:math.RT/0409338, September 2004.
- [GR04b] S. Griffeth and A. Ram, *Affine Hecke algebras and the Schubert calculus*, European J. Combin. **25** (2004), no. 8, 1263–1283.
- [Gr02] W. Graham, *Equivariant K-theory and Schubert varieties*, preprint 2002.
- [Gr01] ———, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614.
- [Ik] T. Ikeda, *Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian*, arXiv:math.AG/0508110.
- [IN] T. Ikeda and H. Naruse, *Excited Young diagrams and equivariant Schubert calculus*, arXiv:math.AG/0703637.
- [KM05] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318.
- [KMY05a] A. Knutson, E. Miller, and A. Yong, *Grobner geometry of vertex decompositions and of flagged tableaux*, preprint, arXiv:math.AG/0502144, February 2005.
- [KMY05b] ———, *Tableau complexes*, preprint, arXiv:math.CO/0510487, October 2005.
- [Kn07] A. Knutson, *Schubert patches degenerate to subword complexes*, preprint, December 2007.
- [KR03] Vijay Kodiyalam and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in Grassmannians*, J. Algebra **270** (2003), no. 1, 28–54.
- [Kra01] C. Krattenthaler, *On multiplicities of points on Schubert varieties in Grassmannians*, Sémin. Lothar. Combin. **45** (2000/01), Art. B45c, 11 pp. (electronic).
- [Kra05] ———, *On multiplicities of points on Schubert varieties in Grassmannians II*, J. Algebraic Combin. **22** (2005), 273–288.
- [KL03] V. Kreiman, *Monomial bases and applications for Richardson and Schubert varieties in ordinary and affine Grassmannians*, Ph.D. thesis, Northeastern University, August 2003.

- [Kre05] ———, *Local properties of Richardson varieties in the Grassmannian via a bounded Robinson-Schensted-Knuth correspondence*, preprint, ArXiv:math.AG/0511695, November 2005.
- [Kre06] ———, *Schubert classes in the equivariant K-theory and equivariant cohomology of the Lagrangian Grassmannian*, preprint, arXiv:math.AG/0602245, February 2006.
- [KT03] Allen Knutson and Terence Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, *Duke Math. J.* **119** (2003), no. 2, 221–260.
- [La90] Alain Lascoux, *Anneau de Grothendieck de la variété de drapeaux*, *The Grothendieck Festschrift, Vol. III*, *Progr. Math.*, vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 1–34.
- [Le04] Cristian Lenart, *A unified approach to combinatorial formulas for Schubert polynomials*, *J. Algebraic Combin.* **20** (2004), no. 3, 263–299.
- [LLT97] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, *Flag varieties and the Yang-Baxter equation*, *Lett. Math. Phys.* **40** (1997), no. 1, 75–90.
- [LRS04] C. Lenart, S. Robinson, and F. Sottile, *Grothendieck polynomials via permutation patterns and chains in the Bruhat order*, preprint, arXiv:math.CO/0405539, May 2004.
- [LRS06] V. Lakshmibai, K. N. Raghavan, and P. Sankaran, *Equivariant Giambelli and determinantal restriction formulas for the Grassmannian*, *Pure Appl. Math. Q.* **2** (2006), no. 3, 699–717.
- [LS82a] Alain Lascoux and Marcel-Paul Schützenberger, *Polynômes de Schubert*, *C. R. Acad. Sci. Paris Sér. I Math.* **294** (1982), no. 13, 447–450.
- [LS82b] ———, *Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux*, *C. R. Acad. Sci. Paris Sér. I Math.* **295** (1982), no. 11, 629–633.
- [Ma91] I. G. Macdonald, *Notes on Schubert polynomials*, *Publications du LACIM, Université du Québec à Montréal*, 1991.
- [Mc06] Peter J. McNamara, *Factorial Grothendieck polynomials*, *Electron. J. Combin.* **13** (2006), no. 1, Research Paper 71, 40 pp. (electronic).
- [MS99] Alexander I. Molev and Bruce E. Sagan, *A Littlewood-Richardson rule for factorial Schur functions*, *Trans. Amer. Math. Soc.* **351** (1999), no. 11, 4429–4443.

- [Wi06] Matthieu Willems, *K-théorie équivariante des tours de Bott. Application à la structure multiplicative de la K-théorie équivariante des variétés de drapeaux*, Duke Math. J. **132** (2006), no. 2, 271–309.

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