Schubert Classes in the Equivariant K-Theory and Equivariant Cohomology of the Lagrangian Grassmannian

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Abstract

We give positive formulas for the restriction of a Schubert Class to a T-fixed point in the equivariant K-theory and equivariant cohomology of the Lagrangian Grassmannian. Our formulas rely on a result of Ghorpade-Raghavan, which gives an equivariant Gröbner degeneration of a Schubert variety in the neighborhood of a T-fixed point of the Lagrangian Grassmannian.

1 Introduction

Let J be the antidiagonal $2n \times 2n$ matrix whose top n antidiagonal entries are 1's and whose bottom n antidiagonal entries are -1's. Then J defines a nondegenerate skew-symmetric inner product on $\mathbb{C}^{2n}$ by $\langle v, w\rangle = v^t J w$, $v, w \in \mathbb{C}^{2n}$. The Lagrangian Grassmannian LGr_n is defined as the set of all n-dimensional complex subspaces V of $\mathbb{C}^{2n}$ which are isotropic under this inner product, i.e., such that for every $v, w \in V$, $\langle v, w\rangle = 0$. The symplectic group $G = Sp_{2n}(\mathbb{C})$ consists of the invertible $2n \times 2n$ complex matrices which preserve this inner product. Let T and B denote the diagonal and upper triangular matrices of G respectively. The natural action of G on LGr_n is transitive and has a unique B-fixed point e_{id}. Thus LGr_n can be identified with G/P_n, where $P_n \supset B$ is the stabilizer of e_{id}. Let W denote the Weyl group of G with respect to T $(= N_G(T)/T)$ and W_{P_n} the Weyl group of P_n. For the G-action on LGr_n, the T-fixed points are precisely the cosets $e_\beta := \beta P_n$, $\beta \in W/W_{P_n}$.

Let B^- denote the lower triangular matrices in G. For $\alpha \in W/W_{P_n}$, the (opposite) Schubert variety X_α is the Zariski closure of $B^- e_\alpha$ in LGr_n. The Schubert variety X_α defines classes $[X_\alpha]_{\mathrm{K}}$ in $K_T^*(LGr_n)$, the T-equivariant K-theory of LGr_n, and $[X_\alpha]_{\mathrm{H}}$ in $H_T^*(LGr_n)$, the T-equivariant cohomology of LGr_n.

The T-equivariant embedding $e_\beta \xrightarrow{i} LGr_n$ induces restriction homomorphisms:

$$K_T^*(LGr_n) \xrightarrow{i_{\mathrm{K}}^*} K_T^*(e_\beta) \qquad \text{and} \qquad H_T^*(LGr_n) \xrightarrow{i_{\mathrm{H}}^*} H_T^*(e_\beta).$$

The image of an element C of $K_T^*(LGr_n)$ or $H_T^*(LGr_n)$ under restriction to e_β is denoted by $C|_{e_\beta}$. The restrictions $C|_{e_\beta}$, evaluated at all $\beta \in W/W_{P_n}$, determine C uniquely. In this paper we obtain combinatorial formulas for $[X_\alpha]_{\text{K}}|_{e_\beta}$ and $[X_\alpha]_{\text{H}}|_{e_\beta}$. Our formula for $[X_\alpha]_{\text{K}}|_{e_\beta}$ is positive in the sense of [4, Conjecture 5.1], and our formula for $[X_\alpha]_{\text{H}}|_{e_\beta}$ is positive in the sense of [5]. A positive formula for $[X_\alpha]_{\text{K}}|_{e_\beta}$ also appears in [15], and positive formulas for $[X_\alpha]_{\text{H}}|_{e_\beta}$ appear in [1] and [6].

The proof of our formulas relies on a result of Ghorpade-Raghavan [3], which gives an explicit equivariant Gröbner degeneration of an open neighborhood of X_α centered at e_β to a reduced union of coordinate spaces. The outline of our proof is virtually the same as that of [14], which derives a similar result as here, but for Schubert varieties in the ordinary Grassmannian. In addition, many of the lemmas of [14] and their proofs carry over with little or no modification.

Our formulas for $[X_\alpha]_{\text{K}}|_{e_\beta}$ and $[X_\alpha]_{\text{H}}|_{e_\beta}$ are expressed in terms of 'semistandard set-valued *shifted* tableaux'. These objects take the place of the 'semistandard set-valued tableaux' in [14]. Semistandard set-valued tableaux were introduced by Buch [2], and also appear in [7], [8]. The formula for $[X_\alpha]_{\text{H}}|_{e_\beta}$ can also be expressed in terms of 'subsets of *shifted* diagrams', which we introduce in Section 5. These objects take the place of the 'subsets of Young diagrams' in [14]. It has come to our attention that Ikeda-Naruse have independently discovered subsets of Young diagrams and subsets of shifted diagrams and used them to express formulas for restrictions of Schubert classes to T-fixed points in the equivariant cohomology of the ordinary and Lagrangian Grassmannians respectively.

2 Semistandard Set-Valued Shifted Tableaux

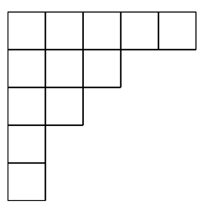
For $k \in \{1, \ldots, 2n\}$, define $\overline{k} = 2n + 1 - k$. Let I_n denote the set of all n element subsets $\alpha = \{\alpha(1), \ldots, \alpha(n)\}$ of $\{1, \ldots, 2n\}$ such that for each $k \in \{1, \ldots, 2n\}$, exactly one of k or $\overline{k}$ is in α. We always assume the entries of such a subset are listed in increasing order. For $\alpha \in I_n$, define $\alpha' \in I_n$ by $\alpha' = \{1, \ldots, 2n\} \setminus \alpha = \{\overline{\alpha(n)}, \ldots, \overline{\alpha(1)}\}$. The map which takes $\{\alpha(1), \ldots, \alpha(n)\} \in I_n$ to the permutation $(\alpha(1), \ldots, \alpha(n), \overline{\alpha(n)}, \ldots, \overline{\alpha(1)}) \in W$ identifies I_n with the set of minimal length coset representatives for W/W_{P_n}. We shall use I_n rather than W/W_{P_n} to index the Schubert varieties and T-fixed points of LGr_n. Fix $\alpha, \beta \in I_n$ for the remainder of this paper.

A **partition** is an ordered list of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_1 \geq \cdots \geq \lambda_m$. Two partitions are identified if one can be obtained from the other by adding zeros. The **transpose** of λ is the partition $\lambda^t = (\lambda_1^t, \ldots, \lambda_p^t)$, where $\lambda_j^t = \#\{i \in \{1, \ldots, m\} \mid \lambda_i \geq j\}$, $j = 1, \ldots, p$. The partition λ is said to be **symmetric** if $\lambda^t = \lambda$. It is said to be **strict** if $\lambda_i = \lambda_{i+1}$ implies $\lambda_i = 0$, $i = 1, \ldots, m$. We denote by L_n (resp. M_n) the set of all symmetric (resp. strict) partitions λ with $\lambda_1 \leq n$.

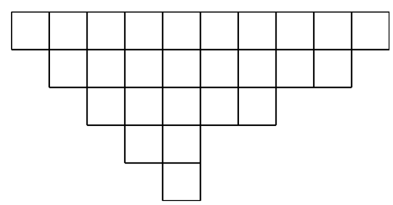
The map π from finite subsets of the positive integers to partitions, given by $\pi : \{\gamma(1), \ldots, \gamma(k)\} \mapsto (\gamma(k) - k, \ldots, \gamma(1) - 1)$, where $\gamma(1) < \cdots < \gamma(k)$, restricts to a bijection from I_n to L_n. The map ρ from partitions to strict partitions, given by $\rho : (\lambda_1, \ldots, \lambda_k) \mapsto (\lambda_1, \lambda_2 - 1, \ldots, \lambda_l - l + 1)$, where l is maximal such that $\lambda_l - l + 1 \geq 0$, restricts to a bijection from L_n to M_n. We denote the composition $\rho \circ \pi : I_n \to M_n$ by σ. If $\lambda = \sigma(\alpha)$, then the **length** of α, denoted $l(\alpha)$, is $\lambda_1 + \cdots + \lambda_n$.

A **Young diagram** is a collection of boxes arranged into a top and left justified array. A Young diagram is said to be **symmetric** if the length of the i-th row equals the length of the i-th column for all i. To any partition λ we associate the Young diagram D_λ whose i-th row has length λ_i. The j-th column of D_λ has length λ^t_j. Thus λ is symmetric if and only if D_λ is symmetric, and L_n can be identified with the set of all symmetric Young diagrams whose first rows have length $\leq n$.

A **shifted diagram** is a top-justified array of boxes whose left side forms a descending staircase, i.e., the leftmost box of any row is one column to the right of the leftmost box of the row above it. The **length** of a row of a shifted diagram is the number of boxes it contains. To a strict partition λ we associate the shifted diagram $\widetilde{D}_\lambda$ whose i-th row has length λ_i. We call λ the **shape** of $\widetilde{D}_\lambda$. One sees that M_n can be identified with the set of all shifted diagrams whose first rows have length $\leq n$.



(a) A symmetric Young diagram



(b) A shifted diagram

The bijection $\rho : L_n \to M_n$ can be viewed in terms of associated partitions. Let λ be a symmetric partition. If we remove all boxes of D_λ which lie below the main diagonal, then we obtain $\widetilde{D}_{\rho(\lambda)}$:

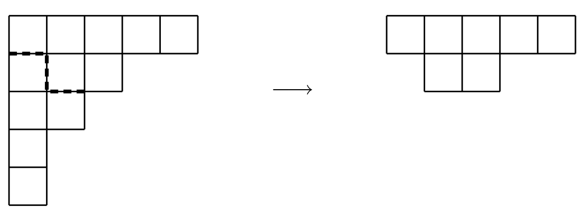


Figure 1: The map ρ

A **set-valued shifted tableau** S is an assignment of a nonempty set of positive integers to each box of a shifted diagram. The entries of S are the positive integers in the boxes. If a positive integer occurs in more than one box of S, then we consider the separate occurrences to be distinct entries. If x is an entry of S, then we define $r(x)$ and $c(x)$ to be the row and column numbers of the box containing x (where the top row is considered the first row and the leftmost column is considered the first column), and we define $z(x)$ to be $x + c(x) - r(x)$. We say that S is a **Young shifted tableau** if each box contains a single entry.

A set-valued shifted tableau is said to be **semistandard** if all entries of any box B are less than or equal to all entries of the box to the right of B and strictly less than all entries of the box below B.

1	2, 3	3	3	4, 6, 7	7, 9
	4	4, 6	6, 7, 8	9, 11	
		8, 10			

Figure 2: A semistandard set-valued shifted tableau

If $\mu = (\mu_1, \ldots, \mu_h)$ is any strict partition, then a set-valued shifted tableau S is said to be **on** μ if, for every entry x of S, $x \leq h$ and

$$z(x) \leq \mu_x + x - 1. \tag{1}$$

Example 2.1. *Let $\lambda = (2, 1)$, $\mu = (5, 3, 2)$. The following list gives all semistandard set-valued shifted tableaux on μ of shape λ:*

1	1
	2

1	1
	3

1	2
	3

2	2
	3

1	1
	2, 3

1	1, 2
	3

1, 2	2
	3

Denote the set of semistandard set-valued shifted tableaux on μ of shape λ by $\mathrm{SSV\widetilde{T}}_{\lambda,\mu}$ and the set of semistandard Young shifted tableaux on μ of shape λ by $\mathrm{SSY\widetilde{T}}_{\lambda,\mu}$.

3 Results

Let $\mathfrak{t}$ denote the Lie algebra of T and $R(T)$ the representation ring of T. We have that

$$T = \{\mathrm{diag}(s_1, \dots, s_n, s_n^{-1}, \dots, s_1^{-1}) \mid s_k \in \mathbb{C}^*\}$$
$$\mathfrak{t} = \{\mathrm{diag}(s_1, \dots, s_n, -s_n, \dots, -s_1) \mid s_k \in \mathbb{C}\}$$

$$K_T^*(e_\beta) \cong R(T) = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$
$$H_T^*(e_\beta) \cong \mathbb{C}[\mathfrak{t}^*] = \mathbb{C}[t_1, \dots, t_n]$$

For $k = 1, \dots, n$, define $t_{\overline{k}} \in K_T^*(e_\beta)$ to be t_k^{-1} and $t_{\overline{k}} \in H_T^*(e_\beta)$ to be $-t_k$.

Proposition 3.1. *Let $\lambda = \sigma(\alpha)$, $\mu = \sigma(\beta)$. Then*
(i) $[X_\alpha]_K|_{e_\beta} = (-1)^{l(\alpha)} \displaystyle\sum_{S \in \mathrm{SSV}\widetilde{\mathrm{T}}_{\lambda,\mu}} \prod_{x \in S} \left(\frac{1}{t_{\beta'(x)} t_{\beta'(z(x))}} - 1 \right).$

(ii) $[X_\alpha]_H|_{e_\beta} = \displaystyle\sum_{S \in \mathrm{SSY}\widetilde{\mathrm{T}}_{\lambda,\mu}} \prod_{x \in S} \left(-t_{\beta'(x)} - t_{\beta'(z(x))} \right).$

Example 3.2. *Consider LGr_3, $\alpha = \{1, 3, \overline{2}\}$, $\beta = \{3, \overline{2}, \overline{1}\}$. Then $\sigma(\alpha) = (2)$, $\sigma(\beta) = (3, 2)$, $l(\alpha) = 2$, $\beta' = \{1, 2, \overline{3}\}$. The semistandard set-valued tableaux on $\sigma(\beta)$ of shape $\sigma(\alpha)$ are:*

1	1

1	2

2	2

1	1, 2

1, 2	2

Therefore,

$$\begin{aligned}
[X_\alpha]_K|_{e_\beta} = & \left(\frac{1}{t_1^2} - 1\right)\left(\frac{1}{t_1 t_2} - 1\right) + \left(\frac{1}{t_1^2} - 1\right)\left(\frac{t_3}{t_2} - 1\right) + \left(\frac{1}{t_2^2} - 1\right)\left(\frac{t_3}{t_2} - 1\right) \\
& + \left(\frac{1}{t_1^2} - 1\right)\left(\frac{1}{t_2^2} - 1\right)\left(\frac{1}{t_1 t_2} - 1\right) + \left(\frac{1}{t_1^2} - 1\right)\left(\frac{1}{t_1 t_2} - 1\right)\left(\frac{t_3}{t_2} - 1\right)
\end{aligned}$$
$$[X_\alpha]_H|_{e_\beta} = (-2t_1)(-t_1 - t_2) + (-2t_1)(-t_2 + t_3) + (-2t_2)(-t_2 + t_3).$$

Remark 3.3. As we shall show in Section 4, each term in the products of Proposition 3.1(i) and (ii) is of the form $e^\theta - 1$ and θ respectively, where θ is a positive root with respect to the Borel subgroup B^-.

4 The Class of a Schubert Variety

The **Plücker map** $LGr_n \to \mathbb{P}(\wedge^n \mathbb{C}^{2n})$ is defined by $V \mapsto [v_1 \wedge \cdots \wedge v_n]$, where $\{v_1, \ldots, v_n\}$ is any basis for V. The Plücker map is a closed immersion, giving LGr_n its projective variety structure.

Reduction to an Affine Variety

Under the Plücker map, e_β maps to $[e_{\beta(1)} \wedge \cdots \wedge e_{\beta(n)}] \in \mathbb{P}(\wedge^n \mathbb{C}^{2n})$. Define p_β to be the homogeneous (**Plücker**) coordinate $[e_{\beta(1)} \wedge \cdots \wedge e_{\beta(n)}]^* \in \mathbb{C}[\mathbb{P}(\wedge^n \mathbb{C}^{2n})]$. Let $\mathcal{O}_\beta$ be the distinguished open set of LGr_n defined by $p_\beta \neq 0$. Then $\mathcal{O}_\beta$ is isomorphic to the affine space $\mathbb{C}^{n(n+1)/2}$, with e_β the origin. Indeed, $\mathcal{O}_\beta$ can be identified with the space of $2n \times n$ complex matrices of the form $K \cdot M$, where K and M are defined as follows:

1. K is the $2n \times 2n$ diagonal matrix which has 1's in the diagonal entries of rows $\beta(1), \ldots, \beta(n)$ and rows $n+1, \ldots, 2n$, and -1's in the diagonal entries of all other rows.
2. M is any $2n \times n$ complex matrix for which rows $\beta(1), \ldots, \beta(n)$ form the $n \times n$ identity matrix and rows $\beta'(1), \ldots, \beta'(n)$ form an $n \times n$ antisymmetric (i.e., symmetric about the antidiagonal) matrix.

Under this identification, we index the rows of $\mathcal{O}_\beta$ by $\{1, \ldots, 2n\}$ and the columns by β. We then choose the coordinates of $\mathcal{O}_\beta$ to be the matrix elements y_{ab}, where $a \in \beta'$, $b \in \beta$, and $a \leq \overline{b}$ (note: due to the antisymmetry in the matrices M of 2, each matrix element y_{ab}, where $a \in \beta'$, $b \in \beta$, and $a > \overline{b}$ must be plus or minus one of our chosen coordinates). Thus $\{(a,b) \in \beta' \times \beta \mid a \leq \overline{b}\}$, which we denote by $\mathfrak{R}_\beta$, forms an indexing set for the coordinates of $\mathcal{O}_\beta$.

Example 4.1. *Let $n = 4$, $\beta = \{1, 4, \overline{3}, \overline{2}\}$. Then $\beta' = \{2, 3, \overline{4}, \overline{1}\}$ and*

$$\mathcal{O}_\beta = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -y_{21} & -y_{24} & -y_{2\overline{3}} & -y_{2\overline{2}} \\ -y_{31} & -y_{34} & -y_{3\overline{3}} & -y_{2\overline{3}} \\ 0 & 1 & 0 & 0 \\ y_{\overline{4}\,1} & y_{\overline{4}\,4} & y_{34} & y_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ y_{\overline{1}\,1} & y_{\overline{4}\,1} & y_{31} & y_{21} \end{pmatrix}, y_{ab} \in \mathbb{C} \right\}.$$

The space $\mathcal{O}_\beta$ is T-stable, and for $\mathbf{s} = \operatorname{diag}(s_1, \ldots, s_n, s_n^{-1}, \ldots, s_1^{-1}) \in T$ and coordinate functions $y_{ab} \in \mathbb{C}[\mathcal{O}_\beta]$,

$$\mathbf{s}(y_{ab}) = \frac{s_b}{s_a}\, y_{ab},$$

where $s_{\overline{k}} := s_k^{-1}$, $k = 1, \ldots, n$.

The equivariant embeddings $e_\beta \xrightarrow{j} \mathcal{O}_\beta \xrightarrow{k} LGr_n$ induce homomorphisms

$$K_T^*(LGr_n) \xrightarrow{k^*} K_T^*(\mathcal{O}_\beta) \xrightarrow{j^*} K_T^*(e_\beta).$$

The map j^* is an isomorphism, identifying $K_T^*(\mathcal{O}_\beta)$ with $K_T^*(e_\beta)$. Define $Y_{\alpha,\beta} = X_\alpha \cap \mathcal{O}_\beta$, an affine subvariety of $\mathcal{O}_\beta$. We have

$$[X_\alpha]_{\mathrm{K}}|_{e_\beta} = j^* \circ k^*([X_\alpha]_{\mathrm{K}}) = j^*([k^{-1}X_\alpha]_{\mathrm{K}}) = j^*([Y_{\alpha,\beta}]_{\mathrm{K}}) = [Y_{\alpha,\beta}]_{\mathrm{K}}.$$

Applying analogous arguments for equivariant cohomology, we obtain

$$[X_\alpha]_{\mathrm{H}}|_{e_\beta} = [Y_{\alpha,\beta}]_{\mathrm{H}}.$$

Reduction to a Union of Coordinate Subspaces

Let $\lambda = \sigma(\alpha)$, $\mu = \sigma(\beta)$. Let $\mathrm{SV\widetilde{T}}_{\lambda,\mu}$ denote the set of all set-valued shifted tableaux (not necessarily semistandard) of shape λ on μ. For $S \in \mathrm{SV\widetilde{T}}_{\lambda,\mu}$, define

$$W_S = V(\{y_{\beta'(x),\overline{\beta'(z(x))}} \mid x \in S\}),$$

a coordinate subspace of $\mathcal{O}_\beta$. Define

$$W_{\alpha,\beta} = \bigcup_{P \in \mathrm{SSY\widetilde{T}}_{\lambda,\mu}} W_P.$$

The following lemma, whose proof is a consequence of [3] and appears in Section 5, reduces the proof of Proposition 3.1 to computing the class of a union of coordinate subspaces.

Lemma 4.2. $[Y_{\alpha,\beta}]_K = [W_{\alpha,\beta}]_K$

Proof of Proposition 3.1. (i) The proofs of Lemmas 4.3 and 4.4 and consequently of Proposition 2.2(i) of [14] carry through if the following modifications are made: (a) the word 'tableau' is replaced by 'shifted tableau' in all steps and all required definitions, and (b) $t_{\beta(d+1-x)}$ and $t_{\beta'(x+c(x)-r(x))}$ are replaced by $t_{\overline{\beta'(z(x))}}$ and $t_{\beta'(x)}$ respectively wherever they occur. The latter modification accounts for the difference in the definitions of W_S.

(ii) There is a standard ring homomorphism from $K_T^*(\mathcal{O}_\beta)$ to $H_T^*(\mathcal{O}_\beta)$, the Chern character map, given by $ch : t_i \mapsto e^{-t_i} = 1 - t_i + t_i^2/2 - t_i^3/3 + \cdots$. If $Y \subset \mathcal{O}_\beta$ is a T-stable subvariety, then

$$ch : [Y]_{\mathrm{K}} \mapsto [Y]_{\mathrm{H}} + \text{ higher order terms.}$$

Thus, $[Y_{\alpha,\beta}]_{\mathrm{H}}$ is the lowest order term of

$$(-1)^{l(\alpha)} \sum_{S \in \mathrm{SSV\widetilde{T}}_{\lambda,\mu}} \prod_{x \in S} \left(\frac{1}{e^{-t_{\beta'(x)}} e^{-t_{\overline{\beta'(z(x))}}}} - 1 \right),$$

which equals

$$\sum_{S \in \widetilde{\mathrm{SSYT}}_{\lambda,\mu}} \prod_{x \in S} \left(-t_{\beta'(x)} - t_{\beta'(z(x))}\right).$$

□

Proof of Remark 3.3. Let $\eta = \pi(\beta)$, so that $\mu = \sigma(\beta) = \rho(\pi(\beta)) = \rho(\eta)$. One can show that $\eta_j = \#\{i \in \{1,\dots,n\} \mid \beta'(i) < \beta(n+1-j)\}$, $j = 1,\dots,n$. Therefore

$$i \leq \eta_j \iff \beta'(i) < \beta(n+1-j) \tag{2}$$

We look at one term $-t_{\beta'(x)} - t_{\beta'(z(x))}$ in the product of (ii). Substituting $i = z(x)$ and $j = x$ into (2), we obtain: $z(x) \leq \eta_x \iff \beta'(z(x)) < \beta(n+1-x) = \overline{\beta'(x)} \iff \beta'(x) < \overline{\beta'(z(x))}$. Since S is on μ, x satisfies (1), i.e., $z(x) \leq \mu_x + x - 1 = \eta_x$. Thus $\beta'(x) < \overline{\beta'(z(x))}$. In addition, since $x \leq z(x)$, $\beta'(x) \leq \beta'(z(x))$.

Thus $-t_{\beta'(x)} - t_{\beta'(z(x))}$ is of the form $-t_a - t_b$, $a \leq b$, $a < \overline{b}$, and hence $a \leq n$. Clearly this is a positive root if $b \leq n$. If $b > n$, then letting $c = \overline{b}$, we have $-t_a - t_b = -t_a + t_c$, $a < c \leq n$, which is also a positive root. □

5 Four Equivalent Models: $\widetilde{\mathcal{F}}''_{\lambda,\mu}$, $\widetilde{\mathcal{F}}_{\lambda,\mu}$, $\widetilde{\mathcal{D}}_{\lambda,\mu}$, and $\widetilde{\mathrm{SSYT}}_{\lambda,\mu}$

In this section we prove Lemma 4.2. We assume all definitions from Sections 5 and 6 of [14]. Let ζ, η be symmetric partitions with $\zeta \leq \eta$. Let D_η be the (symmetric) Young diagram associated to η.

- A family F of nonintersecting paths on D_η is said to be **symmetric** if $(i,j) \in F \iff (j,i) \in F$. In such case, it can be checked inductively that for any path p of F, $\overline{p}$ is also a path of F, where $\overline{p}$ is the path obtained by replacing each (i,j) of p by (j,i). We define $\overline{\mathcal{F}}_{\zeta,\eta}$ and $\overline{\mathcal{F}''}_{\zeta,\eta}$ to be the set of all symmetric elements of $\mathcal{F}_{\zeta,\eta}$ and $\mathcal{F}''_{\zeta,\eta}$ respectively.

- A subset D of D_η is said to be **symmetric** if $(i,j) \in D \iff (j,i) \in D$. We define $\overline{\mathcal{D}}_{\zeta,\eta}$ to be the set of all symmetric elements of $\mathcal{D}_{\zeta,\eta}$.

- A semistandard tableau P of shape ζ is said to be **symmetric** if $P_{i,j} - i = P_{j,i} - j$ for all $(i,j) \in D_\eta$. We define $\overline{\mathrm{SSYT}}_{\zeta,\eta}$ to be the set of all symmetric elements of $\mathrm{SSYT}_{\zeta,\eta}$.

By Lemma 5.14 of [14], $\mathcal{F}''_{\zeta,\eta} = \mathcal{F}_{\zeta,\eta}$. Hence $\overline{\mathcal{F}}''_{\zeta,\eta} = \overline{\mathcal{F}}_{\zeta,\eta}$. The bijections $\mathcal{F}_{\zeta,\eta} \to \mathcal{D}_{\zeta,\eta}$ and $\mathcal{D}_{\zeta,\eta} \to \mathrm{SSYT}_{\zeta,\eta}$ given in [14] restrict to bijections $\overline{\mathcal{F}}_{\zeta,\eta} \to \overline{\mathcal{D}}_{\zeta,\eta}$ and $\overline{\mathcal{D}}_{\zeta,\eta} \to \overline{\mathrm{SSYT}}_{\zeta,\eta}$ respectively.

Let $\lambda = \rho(\zeta)$, $\mu = \rho(\eta)$, and let $\widetilde{D}_\mu$ be the shifted diagram associated with μ. We have the notions of subsets of $\widetilde{D}_\mu$ and families of nonintersecting paths on $\widetilde{D}_\mu$, defined analogously as in [14]. If $D \in \overline{\mathcal{D}}_{\zeta,\eta}$, then we define $\rho(D)$ to be the subset of $\widetilde{D}_\mu$ obtained by removing all boxes of D below the main diagonal of D. If $F \in \overline{\mathcal{F}}_{\zeta,\eta} = \overline{\mathcal{F}}''_{\zeta,\eta}$, then we define $\rho(F)$ to be the family of nonintersecting paths on $\widetilde{D}_\mu$ obtained by removing all boxes in all paths of F below the main diagonal of F. If $P \in \mathrm{SSY\overline{T}}_{\zeta,\eta}$, then we define $\rho(P)$ to be the semistandard shifted tableau obtained by removing all boxes of P below the main diagonal and their entries. Define $\widetilde{\mathcal{D}}_{\lambda,\mu} = \rho(\overline{\mathcal{D}}_{\zeta,\eta})$, $\widetilde{\mathcal{F}}_{\lambda,\mu} = \rho(\overline{\mathcal{F}}_{\zeta,\eta})$, $\widetilde{\mathcal{F}}''_{\lambda,\mu} = \rho(\overline{\mathcal{F}}''_{\zeta,\eta})$, and note $\mathrm{SSY\widetilde{T}}_{\lambda,\mu} = \rho(\mathrm{SSY\overline{T}}_{\zeta,\eta})$.

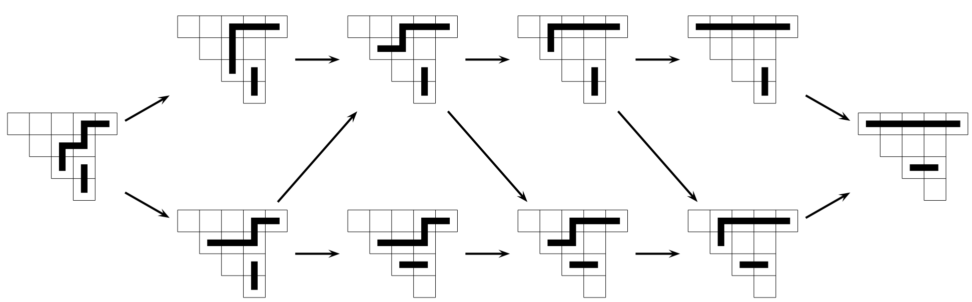
We have that $\widetilde{\mathcal{F}}''_{\lambda,\mu} = \widetilde{\mathcal{F}}_{\lambda,\mu}$, and under ρ, the bijections $\overline{\mathcal{F}}_{\zeta,\eta} \to \overline{\mathcal{D}}_{\zeta,\eta}$ and $\overline{\mathcal{D}}_{\zeta,\eta} \to \mathrm{SSY\overline{T}}_{\zeta,\eta}$ induce bijections $\widetilde{\mathcal{F}}_{\lambda,\mu} \to \widetilde{\mathcal{D}}_{\lambda,\mu}$ and $\widetilde{\mathcal{D}}_{\lambda,\mu} \to \mathrm{SSY\widetilde{T}}_{\lambda,\mu}$ respectively. The following diagram, all of whose squares commute, summarizes our constructions:

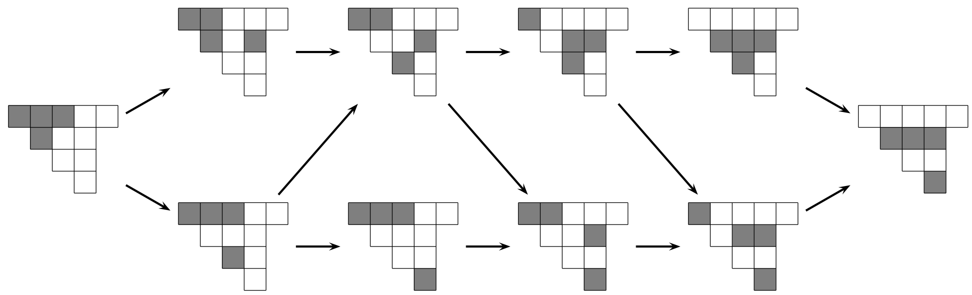
$$
\begin{array}{ccccccc}
\mathcal{F}''_{\zeta,\eta} & = & \mathcal{F}_{\zeta,\eta} & \longrightarrow & \mathcal{D}_{\zeta,\eta} & \longrightarrow & \mathrm{SSYT}_{\zeta,\eta} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{\mathcal{F}}''_{\zeta,\eta} & = & \overline{\mathcal{F}}_{\zeta,\eta} & \longrightarrow & \overline{\mathcal{D}}_{\zeta,\eta} & \longrightarrow & \mathrm{SSY\overline{T}}_{\zeta,\eta} \\
\rho\downarrow & & \rho\downarrow & & \downarrow\rho & & \downarrow\rho \\
\widetilde{\mathcal{F}}''_{\lambda,\mu} & = & \widetilde{\mathcal{F}}_{\lambda,\mu} & \longrightarrow & \widetilde{\mathcal{D}}_{\lambda,\mu} & \longrightarrow & \mathrm{SSY\widetilde{T}}_{\lambda,\mu}
\end{array}
$$

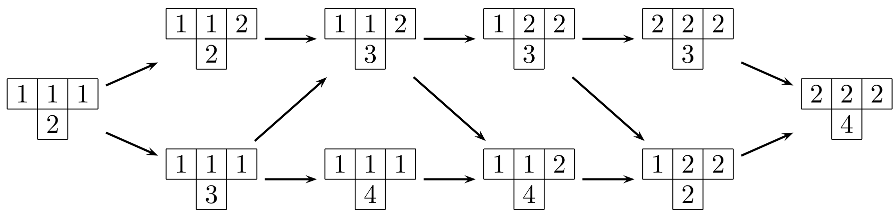
All horizontal maps are bijections, as are the four lower vertical maps. Here we are interested in the bottom row, which gives four equivalent combinatorial models.

The families $\mathcal{F}''_{\zeta,\eta}$ appear in [9], [10], [11], [12], and [13]; $\mathcal{F}''_{\zeta,\eta}$, $\mathcal{F}_{\zeta,\eta}$, $\mathcal{D}_{\zeta,\eta}$, and $\mathrm{SSYT}_{\zeta,\eta}$ appear in [14]; $\overline{\mathcal{F}}''_{\zeta,\eta}$ and $\widetilde{\mathcal{F}}''_{\lambda,\mu}$ were introduced in [3]; $\mathcal{D}_{\zeta,\eta}$, $\overline{\mathcal{D}}_{\zeta,\eta}$, and $\widetilde{\mathcal{D}}_{\lambda,\mu}$ were discovered independently by Ikeda-Naruse.

Example 5.1. *Let* $\lambda = (3,1)$, $\mu = (5,3,2,1)$. *Below we give all elements of* $\widetilde{\mathcal{F}}_{\lambda,\mu}$, $\widetilde{\mathcal{D}}_{\lambda,\mu}$, *and* $\mathrm{SSY\widetilde{T}}_{\lambda,\mu}$.







Proof of Lemma 4.2. Let $\lambda = \sigma(\alpha)$, $\mu = \sigma(\beta)$, $\eta = \pi(\beta)$. Recall that the coordinates $y_{a,b}$ on $\mathcal{O}_\beta$ are indexed by $\mathfrak{R}_\beta = \{(a,b) \in \beta' \times \beta \mid a \leq \overline{b}\}$. Let $\{v_{a,b} \mid (a,b) \in \mathfrak{R}_\beta\} \subset \mathcal{O}_\beta$ denote the basis dual to the basis of linear forms $\{y_{a,b} \mid (a,b) \in \mathfrak{R}_\beta\} \subset \mathcal{O}_\beta^*$. For $F \in \widetilde{\mathcal{F}}''_{\lambda,\mu}$, define

$$W_F = \mathrm{Span}(\{v_{\beta'(x),\beta(n+1-z)} \mid (x,z) \in \mathrm{Supp}(F)\} \,\dot{\cup}\, \{v_{a,b} \mid (a,b) \in \mathfrak{R}_\beta, a > b\}).$$

In [3], an explicit equivariant bijection is constructed from $\mathbb{C}\left[\bigcup_{F \in \widetilde{\mathcal{F}}''_{\lambda,\mu}} W_F\right]$ to $\mathbb{C}[Y_{\alpha,\beta}]$. Thus

$$\mathrm{Char}(\mathbb{C}[Y_{\alpha,\beta}]) = \mathrm{Char}\left(\mathbb{C}\left[\bigcup_{F \in \widetilde{\mathcal{F}}''_{\lambda,\mu}} W_F\right]\right) = \mathrm{Char}\left(\mathbb{C}\left[\bigcup_{F \in \widetilde{\mathcal{F}}_{\lambda,\mu}} W_F\right]\right). \quad (3)$$

Let $\widetilde{D}_\mu$ be the shifted diagram associated to μ. For $x, z \in \{1, \ldots, n\}$, $x \leq z$, we have that $(x,z) \in \widetilde{D}_\mu \iff z \leq \eta_x \iff \beta'(z) < \overline{\beta'(x)} \iff \beta'(x) < \overline{\beta'(z)} = \beta(n+1-z)$ (see proof of Remark 3.3). Thus $\{(a,b) \in \mathfrak{R}_\beta \mid a < b\}$ can be expressed as $\{(\beta'(x), \beta(n+1-z)) \mid (x,z) \in \widetilde{D}_\mu\}$. Let $F \in \widetilde{\mathcal{F}}_{\lambda,\mu}$, and let $D \in \widetilde{\mathcal{D}}_{\lambda,\mu}$ and $P \in \mathrm{SSY\widetilde{T}}_{\lambda,\mu}$ correspond to F under the bijections above. Since $\mathrm{Supp}(F)$ and D are complements in $\widetilde{D}_\mu$,

$$\begin{aligned}\mathfrak{R}_\beta = \{(\beta'(x), \beta(n+1-z)) \mid (x,z) \in \mathrm{Supp}(F)\} \,\dot{\cup}\, \{(a,b) \in \mathfrak{R}_\beta \mid a > b\} \\ \dot{\cup}\, \{(\beta'(x), \beta(n+1-z)) \mid (x,z) \in D\}.\end{aligned}$$

Therefore

$$\begin{aligned}W_F &= V(\{y_{\beta'(x),\beta(n+1-z)} \mid (x,z) \in D\}) \\ &= V(\{y_{\beta'(x),\beta(n+1-x+r(x)-c(x))} \mid x \in P\}) \\ &= V(\{y_{\beta'(x),\beta(n+1-z(x))} \mid x \in P\}) \\ &= V(\{y_{\beta'(x),\overline{\beta'(z(x))}} \mid x \in P\}) \\ &= W_P.\end{aligned}$$

Consequently, $\bigcup_{F\in\widetilde{\mathcal{F}}_{\lambda,\mu}} W_F = \bigcup_{P\in \mathrm{SSY}\widetilde{\mathrm{T}}_{\lambda,\mu}} W_P$, and thus

$$\mathrm{Char}\left(\mathbb{C}\left[\bigcup_{F\in\widetilde{\mathcal{F}}_{\lambda,\mu}} W_F\right]\right) = \mathrm{Char}\left(\mathbb{C}\left[\bigcup_{P\in \mathrm{SSY}\widetilde{\mathrm{T}}_{\lambda,\mu}} W_P\right]\right) = \mathrm{Char}(\mathbb{C}[W_{\alpha,\beta}]). \tag{4}$$

Combining (3) and (4), we obtain $\mathrm{Char}(\mathbb{C}[Y_{\alpha,\beta}]) = \mathrm{Char}(\mathbb{C}[W_{\alpha,\beta}])$. By (4) of [14], $[Y_{\alpha,\beta}]_{\mathrm{K}} = [W_{\alpha,\beta}]_{\mathrm{K}}$. □

References

[1] S. Billey, *Kostant polynomials and the cohomology ring for G/B*, Proc. Nat. Acad. Sci. U.S.A. **94** (1997), no. 1, 29–32.

[2] A. Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78.

[3] S. Ghorpade and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in the symplectic Grassmannian*, preprint, arXiv:math.RT/0409338.

[4] S. Griffeth and A. Ram, *Affine Hecke algebras and the Schubert calculus*, European J. Combin. **25** (2004), no. 8, 1263–1283.

[5] W. Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614.

[6] T. Ikeda, *Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian*, preprint, arXiv:math.AG/0508110.

[7] A. Knutson, E. Miller, and A. Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, preprint, arXiv:math.AG/0502144.

[8] ———, *Tableau complexes*, preprint, arXiv:math.CO/0510487.

[9] V. Kodiyalam and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in Grassmannians*, J. Algebra **270** (2003), no. 1, 28–54.

[10] C. Krattenthaler, *On multiplicities of points on Schubert varieties in Grassmannians*, Sém. Lothar. Combin. **45** (2000/01), Art. B45c, 11 pp. (electronic).

[11] ______, *On multiplicities of points on Schubert varieties in Grassmannians II*, J. Algebraic Combin. **22** (2005), 273–288.

[12] V. Kreiman, *Monomial bases and applications for Richardson and Schubert varieties in ordinary and affine Grassmannians*, Ph.D. thesis, Northeastern University, 2003.

[13] ______, *Local properties of Richardson varieties in the Grassmannian via a bounded Robinson-Schensted-Knuth correspondence*, preprint, arXiv:math.AG/0511695.

[14] ______, *Schubert classes in the equivariant K-theory and equivariant cohomology of the Grassmannian*, preprint, arXiv:math.AG/0512204.

[15] M. Willems, *K-théorie équivariante des variétés de drapeaux et des variétés de Bott-Samelson*, preprint, arXiv:math.AG/0204265.

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