# ON IDEAL GENERATORS FOR AFFINE SCHUBERT VARIETIES

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Abstract. We consider a certain class of Schubert varieties of the affine Grassmannian of type A. By embedding a Schubert variety into a finite-dimensional Grassmannian, we construct an explicit basis of sections of the basic line bundle by restricting certain Plücker co-ordinates.

As a consequence, we write an explicit set of generators for the degree-one part of the ideal of the finite-dimensional embedding. This in turn gives a set of generators for the degree-one part of the ideal defining the affine Grassmannian inside the infinite Grassmannian which we conjecture to be a complete set of ideal generators.

We apply our results to the orbit closures of nilpotent matrices. We describe (in a characteristic-free way) a filtration for the coordinate ring of a nilpotent orbit closure and state a conjecture on the SL(n)-module structures of the constituents of this filtration.

## 1. Introduction

Let $K$ be the base field, which we shall suppose to be algebraically closed of arbitrary characteristic. Let $F := K((t)), A := K[[t]]$. Let $\mathcal{G} = SL_n(F), \mathcal{P} = SL_n(A)$, so that $\mathcal{G}/\mathcal{P}$ is the affine Grassmannian, and let $X$ be an (affine) Schubert variety in $\mathcal{G}/\mathcal{P}$. In our previous paper [12], we constructed an explicit basis for the affine Demazure module $H^0(X, L)$, the space of sections of the basic line bundle $L$ on $\mathcal{G}/\mathcal{P}$, restricted to $X$. As a consequence, we obtained a basis for $H^0(X, L)$ in terms of certain Plücker co-ordinates.

In this paper, a sequel to [12], we first give a different construction of our basis of $H^0(X, L)$ for a certain class of affine Schubert varieties. Our key tool is the matrix presentation of the elements of $\mathcal{G}/\mathcal{P}$, similar to that for the classical Grassmannian. This is constructed by embedding $\mathcal{G}/\mathcal{P}$ inside an infinite Grassmannian, which in turn embeds each affine Schubert variety $X$ inside a certain finite-dimensional Grassmannian.

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After defining the basis elements of $H^0(X, L)$ as restrictions of certain "admissible" Plücker co-ordinates (Definition 4.3.1), we prove that they form a spanning set using a system of degree-one straightening relations, the "shuffles" (Corollary 4.3.3). We prove linear independence inductively by writing certain degree-two relations among the admissible Plücker coordinates and restricting to a smaller $X$ (Section 4.6). Both the shuffles and the degree-two relations follow from the matrix presentation.

As an important application, we obtain explicit generators for the degree-one part of the ideal for our class of affine Schubert varieties inside the finite-dimensional Grassmannian. As a corollary, we obtain generators for the degree-one part of the ideal of the affine Grassmannian inside the infinite Grassmannian (Theorem 4.6.5); and we conjecture that these in fact give a complete set of generators for the ideal. Such ideal generators serve as effective tools in solving the geometric problems, especially in the study of the singularities.

It should be remarked that though ideal generators for affine Schubert varieties (as subvarieties of the affine Grassmannian) may be found using Littelmann's standard monomial basis [16], they are hard to compute (since Littelmann's standard monomial basis is hard to compute). Our goal is to develop a standard monomial theory for affine Schubert varieties in terms of Plücker co-ordinates, in the spirit of the classical work of Hodge ([7, 8]; see also [23]). Construction of a basis for the higher-level Demazure modules $H^0(X, L^{\otimes \ell})$ will be taken up in a subsequent paper.

Another application is to nilpotent orbit closures. Let $\mathcal{N}$ denote the set of all nilpotent matrices in $M_{n\times n}(K)$, a closed affine subvariety of $M_{n\times n}(K)$. The group $\mathrm{GL}_n(K)$ acts on $\mathcal{N}$ by conjugation. Each orbit contains precisely one matrix in Jordan canonical form (up to order of the Jordan blocks). Thus the orbits are indexed by partitions $\mu$ of $n$, i.e. $\mu = (\mu_1, \ldots, \mu_n)$, $n \geq \mu_1 \geq \cdots \geq \mu_n \geq 0$, $\mu_1 + \cdots + \mu_n = n$. We denote the orbit corresponding to the partition $\mu = (\mu_1, \ldots, \mu_n)$ by $\mathcal{N}^0_\mu$, and its closure by $\mathcal{N}_\mu$. Lusztig has shown (cf. [17], [18], [20]) that each $\mathcal{N}_\mu$ is isomorphic to an open subset of a certain affine Schubert variety (generally not of the special class mentioned above).

Using Lusztig's isomorphism (and the matrix representation for the elements of $\mathcal{G}/\mathcal{P}$), we define in a characteristic-free way a filtration $\mathcal{F} := \{\mathcal{F}_{m,\mu}, m \geq 0\}$ for $A_\mu$, the co-ordinate ring of $\mathcal{N}_\mu$: namely, $\mathcal{F}_{m,\mu}$ is the span of monomials in the Plücker co-ordinates on $\mathcal{N}_\mu$ of degree $\leq m$.

**Conjecture:** Let $E = K^n$. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be the partition conjugate to $\mu$. Then there is a characteristic free isomorphism of $SL(E)$-modules

$$\mathcal{F}_{m,\mu} \cong L_{\lambda_1^m} E \otimes \ldots \otimes L_{\lambda_s^m} E$$

where $L_\nu E$ denotes the Weyl module with highest weight $\nu$.

In characteristic zero, this conjecture holds due to [21, 25].

The sections below are organized as follows: In §2, we describe the affine and the infinite Grassmannians, and the embedding of $\mathcal{G}/\mathcal{P}$ inside the infinite Grassmannian. In §3, we describe the Ind-variety structures for the affine and the infinite Grassmannians, and we realize the affine Schubert varieties as closed subvarieties inside certain finite-dimensional Grassmannians. In §4, we describe our special class of affine Schubert varieties, and for $X$ in this class we establish our main results concerning a basis for $H^0(X, L)$ and ideal generators for $X$. We then apply these results to the affine Grassmannian. In §5 we present the results for the nilpotent orbit closures.

## 2. Infinite and affine Grassmannians

In this section, we recall the generalities on loop groups and affine and infinite Grassmannians. For details, we refer to [9, 24] (see also [20]). Let

$$F := K((t)) = \left\{ \sum_{i=-N}^{\infty} a_i t^i \mid a_i \in K \right\}, \quad A := K[[t]] = \left\{ \sum_{i=0}^{\infty} a_i t^i \mid a_i \in K \right\}.$$

If $f = \sum_{i=-N}^{\infty} a_i t^i \in F$, with $a_N \neq 0$, define $\mathrm{ord}(f) := N$, as well as $\mathrm{ord}(0) := -\infty$. Let $n \in \mathbb{N}$ be fixed.

**Definition 2.1.** $\widehat{\mathrm{Gr}}(n)$, the *affine Grassmannian*, is defined to be the set of all $A$-lattices (free $A$-modules of rank $n$) in $F^n$.

Define:

$$K^\infty := \left\{ \sum_{j=N}^{\infty} a_j e_j \mid N \in \mathbb{Z}, \, a_j \in K \right\},$$

and for $k \in \mathbb{Z}$, let:

$$E_k = \left\{ \sum_{j=N}^{\infty} a_j e_j \mid N \geq k, \; a_j \in K \right\}.$$

**Definition 2.2.** $\mathrm{Gr}(\infty)$, the *infinite Grassmannian*, is defined to be the set of subspaces $V \subset K^\infty$ such that $E_m \subset V \subset E_{-m}$, for some $m \in \mathbb{N}$.

Let $e_1, \ldots, e_n$ be the standard basis of $F^n$. Consider the identification of $K$-vector spaces:

$$(*)\qquad \begin{array}{ccc} K^\infty & \simeq & F^n \\ e_{cn+i} & \leftrightarrow & t^c e_i \end{array}$$

for $1 \leq i \leq n$ and $c \in \mathbb{Z}$. We let $t : K^\infty \to K^\infty$ denote the map $e_i \mapsto e_{i+n}$, and we use the same symbol for the map $t : F^n \to F^n$ which is multiplication by $t$ . Indeed, these two maps are identified under $(*)$. Also $\{V \in \mathrm{Gr}(\infty) \mid tV \subset V\}$ gets identified with the set of $t$-stable subspaces of $F^n$, i.e., with the set of $A$-lattices in $F^n$ ; and we obtain an embedding $\widehat{\mathrm{Gr}}(n) \subset \mathrm{Gr}(\infty)$. In particular, $E_1$ corresponds to the standard $A$-lattice $L_1 = \mathrm{span}_A\{e_1, \cdots, e_n\}$.

For $V \in \mathrm{Gr}(\infty)$, define the *virtual dimension of* $V$ as

$$\mathrm{vdim}(V) := \dim(V \,/\, V \cap E_1) - \dim(E_1 \,/\, V \cap E_1)\,.$$

For $j \in \mathbb{Z}$, define $\mathrm{Gr}_j(\infty) := \{V \in \mathrm{Gr}(\infty) \mid \mathrm{vdim}(V) = j\}$ and

$$\widehat{\mathrm{Gr}}_j(n) := \widehat{\mathrm{Gr}}(n) \cap \mathrm{Gr}_j(\infty) = \{V \in \widehat{\mathrm{Gr}}(n) \mid \mathrm{vdim}(V) = j\}\,.$$

We say that a subset $I \subset \mathbb{Z}$ is *almost natural* if $|I \setminus I \cap \mathbb{Z}_+|$ and $|\mathbb{Z}_+ \setminus I \cap \mathbb{Z}_+|$ are both finite numbers, where $\mathbb{Z}_+ = \{i \geq 1\}$. In this case, we define the *virtual cardinality of* $I$ as:

$$\mathrm{vcard}(I) = \|I\| := |I \setminus I \cap \mathbb{Z}_+| - |\mathbb{Z}_+ \setminus I \cap \mathbb{Z}_+|\,.$$

For such $I$, we have the associated coordinate subspace in $\mathrm{Gr}(\infty)$, namely

$$E_I = \left\{ \sum_{i \in I} a_i e_i \mid a_i \in K \right\},$$

and $\mathrm{vdim}(E_I) = \|I\|$.

If $(i_j) = (i_1 < i_2 < \cdots)$ is an increasing sequence of integers, we say that $(i_j)$ is *almost natural* if the set $\{i_j\}$ is almost natural. In this case, we define $\mathrm{vcard}\,(i_j) := \mathrm{vcard}\{i_j\}$. Observe that the increasing sequence $(i_j)$ is almost natural with virtual cardinality $c$ if and only if $i_j = j + c$ for $j \gg 0$. We denote the collection of almost natural sequences by $\mathcal{I}$, and those of virtual cardinality 0 by $\mathcal{I}_0$. We define the *Bruhat order* on increasing sequences by:

$$(i_j) \geq (k_j) \quad \Longleftrightarrow \quad i_j \geq k_j \text{ for all } j\,.$$

Let $S_\infty$ be the group of permutations of $\mathbb{Z}$, and let $\tau \in S_\infty$ be the permutation $\tau(i) = i + n$. Define the *extended affine Weyl group* $\widetilde{W} := \{w \in S_\infty \mid w\tau = \tau w\}$. Note that if $w \in \widetilde{W}$, then $w$ is completely determined by $(w(1), \ldots, w(n))$. Indeed, for $1 \leq i \leq n$, we have $w(nc+i) = w\tau^c(i) = \tau^c w(i)$. Thus we can embed both the finite permutation group $S_n$ and $\mathbb{Z}^n$ in $\widetilde{W}$ as follows. We map $\sigma \in S_n$ to $w \in \widetilde{W}$ such that $w(i) = \sigma(i)$, $1 \le i \le n$; and we map $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ to $w \in \widetilde{W}$ such that $w(i) = \tau^{c_i}(r)$, $1 \le i \le n$. We identify $S_n$ and $\mathbb{Z}^n$ with their images under these embeddings. Then $\mathbb{Z}^n$ is a normal subgroup of $\widetilde{W}$, $\mathbb{Z}^n \cap S_n$ is the identity, and $\mathbb{Z}^n S_n = \widetilde{W}$. That is, $\widetilde{W} = \mathbb{Z}^n \ltimes S_n$, a semidirect product.

If $w \in \widetilde{W}$, then $\{w(j) \mid j \ge 1\}$ is almost natural. Define $I_w$ to be the sequence obtained by listing the elements of $\{w(j) \mid j \ge 1\}$ in increasing order. If we write $w = c\sigma$, where $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ and $\sigma \in S_n$, then we may compute $\|I_w\| = -\sum c_i$. We use $\widetilde{W}^P$ to denote $\mathbb{Z}^n$ identified as a set of coset representatives for $\widetilde{W}/S_n$. One easily sees that a given $w \in \widetilde{W}^P$ is determined by its set $I_w$, and that an almost-natural set $I$ can be realized as $I_w$ whenever $I$ is $\tau$-stable, meaining $\tau I \subset I$ (or equivalently $tE_I \subset E_I$). That is, we have a bijection

$$\begin{array}{rcl} \widetilde{W}^P = \mathbb{Z}^n & \xrightarrow{\sim} & \left\{ I \subset \mathbb{Z} \,\middle|\, \begin{array}{c} I \text{ almost natural} \\ \tau I \subset I \end{array} \right\} \\ w = (c_1, \ldots, c_n) & \mapsto & I_w = \{1{+}c_1 n,\, 2{+}c_2 n, \cdots, n{+}1{+}c_1 n, \ldots\}\,. \end{array}$$

Now define $W := \{w \in \widetilde{W} \mid \|I_w\| = 0\}$. Then $S_n, \mathbb{Z}^n_0 \subset W$, where $\mathbb{Z}^n_0 := \mathbb{Z}^n \cap W = \{c \in \mathbb{Z}^n \mid c_1 + \cdots + c_n = 0\}$, and we have $W = \mathbb{Z}^n_0 \ltimes S_n$, a semidirect product. Note that $W$ is the affine Weyl group associated to $\widehat{\mathrm{SL}}_n = \mathrm{SL}_n(F)$. It is the Coxeter group generated by the adjacent transpositions $s_1, \ldots, s_{n-1} \in S_n$ along with the reflection $s_0$ defined by: $s_0(1) = 0$, $s_0(n) = n{+}1$, and $s_0(i) = i$ for $1 < i < n$.

We use $W^P$ to denote $\mathbb{Z}^n_0$ identified as a set of coset representatives for $W/S_n$. Once again $w \mapsto I_w$ gives a bijection:

$$W^P = \mathbb{Z}^n_0 \xrightarrow{\sim} \{I \in \mathcal{I}_0 \mid \tau I \subset I\}\,.$$

We will denote $E_w := E_{I_w}$.

Now, $\mathcal{G} = \mathrm{GL}_n(K)$ acts transitively on $\widehat{\mathrm{Gr}}(n)$ and the isotropy subgroup at $E_1$ is $\mathcal{P} = \mathrm{GL}_n(A)$, so we have $\widehat{\mathrm{Gr}}(n) \cong \mathcal{G}/\mathcal{P}$. The group $\widetilde{W}$ embed in $\mathcal{G}$ : if $w = c\sigma$, where $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ and $\sigma \in S_n$, we identify $w$ with the matrix $\mathrm{diag}(t^{c_1}, \ldots, t^{c_n}) \cdot [\sigma] \in \mathcal{G}$, where $[\sigma]$ is the permutation matrix associated to $\sigma$. Let $\mathcal{B} = \{g = (g_{ij}) \in \mathrm{GL}_n(A) \mid \mathrm{ord}(g_{ij}) > 0 \text{ for } i < j\}$. We have the Bruhat decomposition:

$$\mathcal{G} = \coprod_{w \in \widetilde{W}^P} \mathcal{B} w \mathcal{P}\,,$$

and projection onto $\mathcal{G}/\mathcal{P}$ gives:

$$\widehat{\mathrm{Gr}}(n) \cong \mathcal{G}/\mathcal{P} = \coprod_{w \in \widetilde{W}^P} \mathcal{B}E_w,$$

where $E_w = w\mathcal{P} \in \mathcal{G}/\mathcal{P}$, or equivalently $E_w = E_{I_w} \in \widehat{\mathrm{Gr}}(n)$. For any $V \in \mathcal{B}E_w$, we have $\mathrm{vdim}(V) = \mathrm{vdim}(E_w) = \|I_w\|$. The orbit $\mathcal{B}E_w \subset \mathcal{G}/\mathcal{P}$ is called the *affine Schubert cell* associated to $w$, and is denoted by $X^\circ(w)$.

Let $\mathcal{G}_0 := \{g \in \mathcal{G} \mid \mathrm{ord}(\det g) = 0\} \supset \mathcal{P}$. Then $\mathcal{G}_0$ acts transitively on $\widehat{\mathrm{Gr}}_0(n)$ and the isotropy subgroup at $E_1$ is $\mathcal{P}$, so $\widehat{\mathrm{Gr}}_0(n) \cong \mathcal{G}_0/\mathcal{P}$. We again have $W \hookrightarrow \mathcal{G}_0$ and the Bruhat decomposition:

$$\mathcal{G}_0 = \coprod_{w \in W^P} \mathcal{B}w\mathcal{P}.$$

Projection onto $\mathcal{G}_0/\mathcal{P}$ gives:

$$\widehat{\mathrm{Gr}}_0(n) \cong \mathcal{G}_0/\mathcal{P} = \coprod_{w \in W^P} \mathcal{B}E_w\,.$$

Henceforth we focus on $\widehat{\mathrm{Gr}}_0(n)$. Note that $\mathrm{SL}_n(F)$ also acts transitively on $\widehat{\mathrm{Gr}}_0(n)$, the isotropy at $E_1$ being $\mathrm{SL}_n(A)$, so $\widehat{\mathrm{Gr}}_0(n) \cong \mathrm{SL}_n(F)/\mathrm{SL}_n(A)$.

We will see in the following section that, although $\widehat{\mathrm{Gr}}_0(n) \subset \mathrm{Gr}_0(\infty)$ are Ind-varieties of infinite dimension, the Schubert cell $X^\circ(w)$ is an ordinary variety with the finite dimension $\sum_{j=1}^{\infty}(j - i_j)$, where $I_w = (i_1{<}i_2{<}\cdots)$ and $i_j = j$ for $j \gg 0$.

## 3. Ind-variety Structures

In this section, we recall the Ind-variety structure for affine and infinite Grassmannians. See [13, 14] for details.

For $s \in \mathbb{N}$, let $V_s = t^{-s(n-1)}L_1/t^sL_1$, with $\dim_K(V_s) = sn^2$. Consider the embedding defined by:

$$\begin{aligned} \phi_s : \wedge^{sn} V_s &\hookrightarrow \wedge^{(s+1)n} V_{s+1} \\ v_1 \wedge \cdots \wedge v_{sn} &\mapsto v_1 \wedge \cdots \wedge v_{sn} \wedge t^s e_1 \wedge \cdots \wedge t^s e_n\,, \end{aligned}$$

and let $\wedge^\infty V_\infty$ be the direct limit vector space $\varinjlim\,(\wedge^{sn} V_s)$.

The map $\phi_s$ induces $\phi_s^\star : \wedge^{(s+1)n}(V^\star)_{s+1} \to \wedge^{sn}(V^\star)_s$, and we define $\wedge^\infty V_\infty^\star$ to be the vector space $\varprojlim\,(\wedge^{sn} V_s^\star)$. There is a bilinear pairing between $\wedge^\infty V_\infty$ and $\wedge^\infty V_\infty^\star$ implied by the universal properties of limits under which $\wedge^\infty V_\infty^\star = (\wedge^\infty V_\infty)^\star$, the dual space of $\wedge^\infty V_\infty$. The universal properties of limits also imply for all $s$ the existence of an injection $i_s : \wedge^{sn} V_s \to \wedge^\infty V_\infty$ and a projection $\pi_s : \wedge^\infty V_\infty^\star \to \wedge^{sn}(V_s^\star) = (\wedge^{sn} V_s)^\star$.

If $I = (i_1, i_2, \ldots) \in \mathcal{I}_0$, we can view $e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots$ as an element of $\wedge^\infty V_\infty$ and $p_I := e_{i_1}^\star \wedge e_{i_2}^\star \wedge \cdots$ as an element of $\wedge^\infty V_\infty^\star$. We call the $p_I$ the *infinite Plücker coordinates*. Indeed $\{e_I \mid I \in \mathcal{I}_0\}$ forms a basis for $\wedge^\infty V_\infty$ and $\{p_I \mid I \in \mathcal{I}_0\}$ is the dual basis for $\wedge^\infty V_\infty^\star$, i.e., $\langle e_I, p_J \rangle = \delta_{IJ}$ (Kronecker delta) for $I, J \in \mathcal{I}_0$.

The map $\phi_s$ descends to a closed immersion $\mathbb{P}(\wedge^{sn} V_s) \to \mathbb{P}(\wedge^{(s+1)n} V_{s+1})$, and we let $\mathbb{P}(\wedge^\infty V_\infty) = \bigcup_{s \in \mathbb{N}} \mathbb{P}(\wedge^{sn} V_s)$ with the reduced projective Ind-variety structure (see [13], [14]). The Ind-variety homogeneous coordinate ring of $\mathbb{P}(\wedge^\infty V_\infty)$ is defined to be $K[\mathbb{P}(\wedge^\infty V_\infty)] := \varprojlim K[\wedge^{sn} V_s] = \mathrm{Sym}(\wedge^\infty V_\infty^\star)$.

Fix an integer $s \geq 0$, and let

$$\mathcal{F}_s = \left\{ V \in \mathrm{Gr}_0(\infty) \;\middle|\; \begin{array}{c} t^{-s(n+1)} E_1 \supset V \supset t^s E_1 \\ \dim_K(V / t^s E_1) = sn \end{array} \right\}.$$

We identify $\mathcal{F}_s$ with its image under the bijection $b_s : \mathcal{F}_s \to \mathrm{Gr}(sn, V_s)$, $V \mapsto V / t^s E_1$. Now $\mathrm{Gr}(sn, V_s)$ embeds as a closed subvariety of $\mathbb{P}(\wedge^{sn} V_s)$ under the Plücker embedding $j_s$, and we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Gr}(sn, V_s) & \xrightarrow{j_s} & \mathbb{P}(\wedge^{sn} V_s) \\ \big\downarrow & & \big\downarrow \phi_s \\ \mathrm{Gr}((s+1)n, V_{s+1}) & \xrightarrow{j_{s+1}} & \mathbb{P}(\wedge^{(s+1)n} V_{s+1}) . \end{array}$$

Thus $\bigcup\limits_{s \geq 0} \mathcal{F}_s \cong \bigcup_{s \geq 0} \mathrm{Gr}(sn, V_s)$ induces the structure of a closed Ind-subvariety of $\mathbb{P}(\wedge^\infty V_\infty)$.

We claim that $\mathrm{Gr}_0(\infty) = \bigcup_{s \geq 0} \mathcal{F}_s$. Indeed, for $V \in \mathrm{Gr}(\infty)$ with $t^{-s(n+1)} E_1 \supset V \supset t^s E_1$, we have $V \in \mathcal{F}_s$ if and only if

$$\begin{aligned} sn &= \dim(V / t^s E_1) \\ &= \dim(V \,/\, V \cap E_1) + \dim(V \cap E_1 \,/\, t^s E_1) \\ &= \dim(V \,/\, V \cap E_1) + (\dim(E_1 / t^s E_1) - \dim(E_1 \,/\, V \cap E_1)) \\ &= \dim(V \,/\, V \cap E_1) + sn - \dim(E_1 \,/\, V \cap E_1), \end{aligned}$$

i.e., whenever $\mathrm{vdim}(V) = 0$, meaning $V \in \mathrm{Gr}_0(\infty)$.

Multiplication by $t$ induces a nilpotent endomorphism $t_s$ on each $V_s$. Define $u_s := 1 + t_s$, a unipotent automorphism of $V_s$, and denote the induced automorphism of $\mathrm{Gr}(sn, V_s)$ also by $u_s$. Let

$$\mathcal{H}_s = \left\{ V \in \widehat{\mathrm{Gr}}_0(n) \;\middle|\; \begin{array}{c} t^{-s(n-1)} L_1 \supset V \supset t^s L_1 \\ \dim_K(V / t^s L_1) = sn \end{array} \right\}.$$

Using the identification $(*)$ of Section 2 and the isomorphism $b_s$ above, we have:

$$\mathcal{H}_s \overset{(*)}{\cong} \{V \in \mathcal{F}_s \mid tV \subset V\} \overset{b_s}{\cong} \mathrm{Gr}(sn, V_s)^{u_s},$$

a closed projective subvariety of $\mathrm{Gr}(sn, V_s)$. Therefore $\widehat{\mathrm{Gr}}_0(n) = \bigcup_{s\geq 0} \mathcal{H}_s \cong \bigcup_{s\geq 0} \mathrm{Gr}(sn, V_s)^{u_s}$ induces the structure of a closed Ind-subvariety of $\mathrm{Gr}_0(\infty)$. For $w \in W^P$, we define the *affine Schubert variety* $X(w)$ to be the Zariski closure $\overline{X^\circ(w)} = \overline{\mathcal{B}E_w} \subset \widehat{\mathrm{Gr}}_0(n)$.

The following diagram illustrates the relationships between the various varieties and Ind-varieties:

$$(1)\qquad \begin{array}{ccccccc}
X(w) & \longrightarrow & \widehat{\mathrm{Gr}}_0(n) & \longrightarrow & \mathrm{Gr}_0(\infty) & \longrightarrow & \mathbb{P}(\wedge^\infty V_\infty) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathrm{Gr}(sn, V_s)^{u_s} & \longrightarrow & \mathrm{Gr}(sn, V_s) & \longrightarrow & \mathbb{P}(\wedge^{sn} V_s)
\end{array}$$

If we consider each projective variety in the bottom row as a closed Ind-subvariety of the projective Ind-variety above it, all maps in the diagram are closed immersions of Ind-varieties.

Thus we have the corresponding graded projections (i.e. restrictions) of the homogeneous coordinate rings:

$$(2)\qquad \begin{array}{ccccccc}
 & & & & & & \mathrm{Sym}(\wedge^\infty V_\infty^\star) \\
 & & & & & & || \\
K[X(w)] & \longleftarrow & K[\widehat{\mathrm{Gr}}_0(n)] & \longleftarrow & K[\mathrm{Gr}_0(\infty)] & \longleftarrow & K[\mathbb{P}(\wedge^\infty V_\infty)] \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & K[\mathrm{Gr}(sn, V_s)^{u_s}] & \longleftarrow & K[\mathrm{Gr}(sn, V_s)] & \longleftarrow & K[\mathbb{P}(\wedge^{sn} V_s)] \\
 & & & & & & || \\
 & & & & & & \mathrm{Sym}((\wedge^{sn} V_s)^\star)
\end{array}$$

Note that the projections of the infinite Plücker coordinates to the third row of (2) are just the usual finite Plücker coordinates.

**Proposition 3.1.** (i) $p_I|_{X(w)} = 0 \iff I \not\leq I_w$

(ii) *For $w \in W^P$, we have $X(w) = \dot{\cup} X^\circ(y)$, the disjoint union being over $\{y \in W^P \mid y \leq w\}$.*

(iii) *For $w_s = (c_1, \ldots, c_n) := (-s(n-1), s, \ldots, s) \in W^P$, we have the isomorphisms of reduced varieties:*

$$X(w_s) = \left\{ A\text{-lattices } L \;\middle|\; \begin{array}{c} t^{-s(n-1)} L_1 \supset L \supset t^s L_1 \\ \dim(L/t^s L_1) = sn \end{array} \right\} \cong \mathrm{Gr}(sn, V_s)^{u_s}.$$

**Remark 3.2.** The Proposition implies: $\widehat{\mathrm{Gr}}_0(n) = \varinjlim X(w_s)$.

We will adopt an abuse of notation when dealing with almost natural sets. For any $I = (i_1, i_2, \cdots) \in \mathcal{I}_0$, there is some $m \geq 0$ such that $i_j = j$ for $j \geq m$. We shall ignore the "trivial" entries $i_j = j$ for $j \geq m$, and write just the $(m-1)$-tuple $I = (i_1, \cdots, i_{m-1})$. Any such sequence may also be considered as an $\ell$-tuple for $\ell \geq m-1$ by restoring some of the entries $i_j = j$. Thus, any $I \in \mathcal{I}_0$ may be thought of as an $sn$-tuple for $s \gg 0$. Now part (ii) of the Proposition implies that every $X(w)$ sits inside $X(w_s)$ for $s \gg 0$.

The action by left translations of $\mathrm{SL}_n(F)$ on $\widehat{\mathrm{Gr}}_0(n)$ induces a projective representation of $\mathrm{SL}_n(F)$ and its Lie algebra $\mathfrak{sl}_n(F)$ on $K[\widehat{\mathrm{Gr}}_0(n)]$. This lifts to an actual representation of a central extension, the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_n$ acting on $K[\widehat{\mathrm{Gr}}_0(n)]$.

**Theorem 3.3.** *(cf.* [14]*)*

(i) *Let $V_{\Lambda_0}$ be the basic integrable irreducible representation of $\widehat{\mathfrak{sl}}_n$. Then $V^\star_{d\Lambda_0} \cong K[\widehat{\mathrm{Gr}}_0(n)]_d = \mathrm{Sym}^d(\wedge^\infty V^\star_\infty)|_{\widehat{\mathrm{Gr}}_0(n)}$ as $\widehat{\mathfrak{sl}}_n$ representations.*

(ii) *For $w \in W^P$ let $V_{\Lambda_0}(w)$ be the affine Demazure module of $V_{\Lambda_0}$ corresponding to $w$. Then*

$$V^\star_{d\Lambda_0}(w) \cong K[X(w)]_d = \mathrm{Sym}^d(\wedge^\infty V^\star_\infty)|_{X(w)}\,.$$

## 4. The Schubert variety $X(w_s)$

4.1. **Renormalization of indices.** Recall the special element

$$w_s = (-s(n-1), s, \ldots, s) \in W^P = \mathbb{Z}_0^n\,,$$

with $I_{w_s} = (1 - s(n-1)n,\ 1 - s(n-1)n + n, \ldots, 1 + sn - n,\ 1 + sn)\,.$

In order to avoid continually writing negative indices, we now renormalize these values. This means that, instead of working with the component $\widehat{\mathrm{Gr}}_0(n)$, we consider the isomorphic Ind-variety

$$\widehat{\mathrm{Gr}}_0'(n) := t^{s(n-1)}\,\widehat{\mathrm{Gr}}_0(n) \ = \ \{V \in \widehat{\mathrm{Gr}}(n) \mid \mathrm{vdim}(V) = -s(n-1)n\}\,.$$

The Bruhat decomposition becomes:

$$\widehat{\mathrm{Gr}}_0'(n) = \bigcup_w X^\circ(w)\,,$$

where the union is over $w$ in:

$$\begin{aligned} \acute{W}^P \ &:= \ \tau^{s(n-1)} W^P = \tau^{s(n-1)}\mathbb{Z}_0^n \\ &= \ \{\ w = (c_1, \ldots, c_n) \mid c_1 + \cdots + c_n = s(n-1)n\}\,, \end{aligned}$$

so that
$$I_w \in \mathcal{I}_0' \ := \ \tau^{s(n-1)}\mathcal{I}_0 \ = \ \mathcal{I}_{-s(n-1)n}\,.$$

In this normalization, the zero-dimensional Schubert variety is $X(w) = \{E_w\}$ for
$$w = \tau^{s(n-1)} = (s(n-1), \ldots, s(n-1))$$
$$I_w = \{1{+}sn(n{-}1), 2{+}sn(n{-}1), 3{+}sn(n{-}1), \ldots\}\,.$$
We renormalize:
$$w_s := (0, sn, \ldots, sn)$$
$$I_{w_s} = (1, n{+}1, 2n{+}1, \cdots, sn^2{+}1, sn^2{+}2, sn^2{+}3, \ldots)\,.$$
Further, we can rephrase Proposition 3.1(iii) as:
$$X(w_s) = \left\{ A\text{-lattices } L \subset F^n \;\middle|\; \begin{array}{c} L_1 \supset L \supset t^{sn}L_1 \\ \dim L/t^{sn}L_1 = sn \end{array} \right\} \cong \mathrm{Gr}(sn, V_s)^{u_s}\,,$$
where $V_s = L_1/t^{sn}L_1$.

4.2. $\mathbb{Z}{\times}\mathbb{Z}_+$ **matrix presentation.** As with the finite-dimensional Grassmannians, an element $V \in \mathrm{Gr}(\infty)$ may be thought of as the column space of a matrix with two-sided infinite columns and one-sided infinite rows: a $\mathbb{Z}{\times}\mathbb{Z}_+$ matrix $M = (a_{ij})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}_+}$. Since $E_j \subset V$ for $j \gg 0$, we write $A$ so that $a_{ij} = \delta_{ij}$ except for finitely many entries. Using the $(*)$ identification $K^\infty \simeq F^n$, $e_{cn+i} \leftrightarrow t^c e_i$, an $A$-lattice in $F^n$ also has a $\mathbb{Z}{\times}\mathbb{Z}_+$ matrix presentation.

For example, we have the coordinate lattice: $E_I = \mathrm{Span}_K(e_i \mid i \in I)$ for $I \in \mathcal{I}$. Writing each basis element $e_i$ as a column vector with a 1 in row $i$ and 0 elsewhere, we obtain the matrix presentation of $E_I$.

Now let us consider $w = w_s$, $I = I_{w_s}$. We set:
$$\boxed{r := sn}\,,$$
so that $X(w_s) \subset \mathrm{Gr}(r, V_s)$ where
$$V_s = \mathrm{Span}_K\{t^c e_1, \ldots, t^c e_n \mid 0 \le c \le r{-}1\}\,.$$
We may write each vector $v \in V_s$ with respect to this basis in the block form:
$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix}$$

where each $v_i$ is an $n\times 1$ column vector. Multiplication by $t$ becomes:

$$t(v) = \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ \vdots \\ v_{r-1} \end{pmatrix}$$

Taking the $rn \times r$ matrix presentation of $\mathrm{Gr}(r, V_s)$, we identify

$$\mathrm{Gr}(r, V_s) \;\cong\; M^{\max}_{rn\times r}(K) \,/\, \mathrm{GL}_r(K)\,,$$

where $M^{\max}$ indicates matrices of maximal rank, and two matrices define the same subspace $V \subset V_s$ if they differ by a column transformation in $\mathrm{GL}_r(K)$. The $rn \times r$ matrix $M$ of a space $V \in \mathrm{Gr}(r, V_s)$ is the "relevant part" of the $\mathbb{Z} \times \mathbb{Z}_+$ matrix corresponding to the lattice $L = V \oplus t^r L_1 \in \widehat{\mathrm{Gr}}(n)$.

For a generic subspace $V$ in the Schubert cell $X^\circ(w_s) \subset \mathrm{Gr}(r, V_s)^{u_s}$, we can find a *generating vector* $v \in V$ such that its $e_1$ component is equal to 1 and $V = \mathrm{Span}_k(v, tv, \ldots, t^{r-1}v)$: indeed, any vector $v = e_1 + a_2 v_2 + \cdots$ is a generating vector. Thus $V$ can be presented by an $rn \times r$ matrix of the form:

$$M = (a_{ij}) = \begin{pmatrix} v & tv & \cdots & t^{r-1}v \end{pmatrix} \;=\; \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{pmatrix},$$

where each $A_k$ is an $n \times r$ block. This is a lower triangular $rn \times r$ matrix, meaning $a_{ij} = 0$ for $i < j$ , and the $1^{\text{st}}$ column of $A_1$ repeats as the $2^{\text{nd}}$ column of $A_2$, the $3^{\text{rd}}$ column of $A_3$, etc.

In particular, we have $1 = a_{1,1} = a_{1+n,2} = \cdots = a_{(r-1)n+1,r}$. We shall refer to these rows $1,\ 1{+}n, \ldots,\ 1{+}(r{-}1)n$ as the *pivotal rows.* Now we may reduce $M$ by column operations so that each $a_{1,1} = a_{n+1,2} = \cdots = 1$ is the *only* nonzero entry in a pivotal row. This normalizes $M$ to make it a unique represenative of $V \in X^\circ(w_s)$.

That is, we may identify $X^\circ(w_s) \subset \mathrm{Gr}(r, V_s)$ with the affine space of matrices $M = (a_{i,j}) \in M_{rn\times r}(K)$ of the form:

$$M = \left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & \cdots & 0 \\
a_{2,1} & 0 & 0 & 0 & \cdots & 0 \\
a_{3,1} & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n,1} & 0 & 0 & 0 & \cdots & 0 \\ \hline
0 & 1 & 0 & 0 & \cdots & 0 \\
a_{n+2,1} & a_{2,1} & 0 & 0 & \cdots & 0 \\
a_{n+3,1} & a_{3,1} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{2n,1} & a_{n,1} & 0 & 0 & \cdots & 0 \\ \hline
0 & 0 & 1 & 0 & \cdots & 0 \\
a_{2n+2,1} & a_{n+2,1} & a_{2,1} & 0 & \cdots & 0 \\
a_{2n+3,1} & a_{n+3,1} & a_{3,1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{3n,1} & a_{2n,1} & a_{n,1} & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \cdots & 1 \\
a_{(r-1)n+2,1} & a_{(r-2)n+2,1} & a_{(r-3)n+2,1} & \cdots & \cdots & a_{2,1} \\
a_{(r-1)n+3,1} & a_{(r-2)n+3,1} & a_{(r-3)n+3,1} & \cdots & \cdots & a_{3,1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{rn,1} & a_{(r-1)n,1} & a_{(r-2)n,1} & \cdots & \cdots & a_{n,1}
\end{array}\right].$$

Now it is clear from inspecting the above matrix that, the infinite Plücker coordinates on $\mathrm{Gr}_0(\infty)$, when restricted to $\mathrm{Gr}(r, V_s)$, either vanish or become usual Plücker coordinates on $\mathrm{Gr}(r, V_s)$, the $r \times r$ minors of the above matrix $M$. Indeed, the coordinate $p_I|_{X(w_s)}$ is non-zero for $I = (i_1 < i_2 < \cdots)$ if and only if

(1) $i_j \geq (j-1)n+1$ for $j = 1, 2, \ldots, r$.
(2) $i_j = j + rn$ for $j \geq r+1$.

We shall denote such an index set as an $r$-tuple $I = (i_1, \ldots, i_r)$.

4.3. **Generation by admissible Plücker Coordinates.**

**Definition 4.3.1.** A set $I = (i_1, i_2, \ldots) \in \mathcal{I}_0'$ is *admissible* if

$$i_{j+1} - i_j \leq n \quad \text{for} \quad j \geq 1\,.$$

In this case we also say that the Plücker coordinate $p_I \in K[\widehat{\mathrm{Gr}}_0'(n)]$ is admissible. If $w \in \acute{W}^P$, then $I$ (or $p_I$) is said to be *admissible* on $X(w)$ if $I$ is admissible and $p_I|_{X(w)} \neq 0$.

We proceed to show that on $X(w_s)$, a non-admissible Plücker coordinate $p_I$ is a linear combination of admissible Plücker coordinates.

**Lemma 4.3.2.** *Let $A, B \in M_{m\times m}(K)$. Fix $\ell \leq m$. Then the sum of all determinants obtained by replacing $\ell$ rows of $A$ by the corresponding $\ell$ rows of $B$ equals the sum of all determinants obtained by replacing $\ell$ columns of $A$ by the corresponding $\ell$ columns of $B$.*

*Proof.* One sees by performing Laplace expansions that (1) the sum of all determinants obtained by replacing $\ell$ rows of $A$ by the corresponding $\ell$ rows of $B$ ; and (2) the sum of all determinants obtained by replacing $\ell$ columns of $A$ by the corresponding $\ell$ columns of $B$ ; are both equal to (3) the sum of all products of $\ell$ minors of $B$ with complementary $(m-\ell)$ minors of $A$. □

Let $I = (i_1, \ldots, i_r)$ be such that $p_I|_{X(w_s)} \neq 0$, so that $i_r \leq rn$. Further let $i_1 \geq n+1$, and let $\widetilde{I} = (i_1-n, \ldots, i_r-n)$. Choose $\ell \leq r$. We define a *shuffle* to be the sum of all Plücker coordinates obtained by replacing $\ell$ elements $J \subset I$ with the corresponding $\ell$ elements $\widetilde{J} \subset \widetilde{I}$:

$$\mathrm{sh}_I := \sum_{\substack{J\subset I\\ |J|=\ell}} p_{I\setminus J\cup \widetilde{J}}\,.$$

**Corollary 4.3.3.** *The shuffles are identically* 0 *as functions on $X(w_s)$:*

$$\mathrm{sh}_I|_{X(w_s)} = 0\,.$$

*Proof.* Let $I, \tilde{I}$ be as above. Let $M_{rn\times r}$ be the normalized matrix corresponding to a generic point of $X^\circ(w_s)$, and let $M$ (resp. $\widetilde{M}$) be the $r\times r$ submatrix of $M_{rn\times r}$ with row indices given by $I$ (resp. $\widetilde{I}$). Then we have that the last $r-1$ columns of $M$ are same as the first $r-1$ columns of $\widetilde{M}$, and the last column of $\widetilde{M}$ consists of zeroes. (Note that since $i_r \leq rn$, we have, $i_r-n \leq (r-1)n$. Hence, the entries of $\widetilde{M}$ are taken from the first $(r-1)$ blocks of $M_{rn\times r}$.)

This implies the vanishing of the minor obtained by replacing any $\ell$ columns of $M$ with the corresponding $\ell$ columns of $\widetilde{M}$. Hence in view of Lemma 4.3.2, we obtain the vanishing of the sum of all minors of $M$ obtained by replacing $\ell$ rows of $M$ with the corresponding $\ell$ rows of $\widetilde{M}$. But this sum is simply the evaluation on $M_{rn\times r}$ of the sum of all Plücker coordinates obtained by replacing $\ell$ indices of $I$ with the corresponding $\ell$ indices of $\widetilde{I}$. The desired result now follows. □

**Proposition 4.3.4.** *Let $I = (i_1, \cdots, i_r)$ be a non-admissible $r$-tuple such that $p_I \,|\, _{X(w_s)} \neq 0$. Then on $X(w_s)$ we have, $p_I = \sum_S a_S p_S$ , $a_S \in K$, where the sum is over admissible $r$-tuples.*

*Proof.* Let $\ell \leq r{-}1$ be the smallest integer such that $i_{\ell+1} - i_\ell > n$. Let

$I_1 = (i_1{+}n, \cdots, i_\ell{+}n, i_{\ell+1}, \cdots, i_r),\ I_2 = (i_1, \cdots, i_\ell, i_{\ell+1}{-}n, \cdots, i_r{-}n)$

Note that the (positive) difference between the respective entries of $I_1, I_2$ equals $n$. Hence by Corollary 4.3.3, the corresponding shuffle equals 0 on $X(w_s)$. Let $p_J$ be a typical term appearing in the shuffle. Corresponding to the switch of $\{i_1{+}n, \cdots, i_\ell{+}n\}$ respectively with $\{i_1, \cdots, i_\ell\}$, we have, $J$ equals $I$, and any other $J$ is lexicographically greater than $I$. Now the result follows by induction: note that the lexicographically largest $r$-tuple is $((n{-}1)r + 1,\ (n{-}1)r + 2, \cdots, rn)$ which corresponds to the identity or zero-dimensional Schubert variety). □

4.4. **The reduced chain of $w_s$.** As an element of the Weyl group $W = \langle s_0, s_1, \ldots, s_{n-1}\rangle$, we can write a reduced expression for $w_s$ as:

$$w_s = \underbrace{(s_1 s_2 \cdots s_{n-1} s_0) \cdots (s_1 s_2 \cdots s_{n-1} s_0)}_{s(n-1)\ \text{times}}$$

(In particular, $\dim_K X(w_s) = s(n{-}1)n = r(n{-}1)$, which is indeed the number of variables in the generic matrix above.)

Consider the full chain of $w$'s obtained by deleting one simple reflection at a time in the above reduced expression for $w_s$ (starting from the left). Let us denote these by $\tau_i$ for $i = 1, \ldots, s(n{-}1)n{+}1 = r(n{-}1){+}1$, so that

$$w_s = \tau_1 > \tau_2 > \cdots > \tau_{r(n-1)+1} = \mathrm{id}\,.$$

Let us collect these into $r$ groups of $n$ elements so that the first group consists of $\{\tau_1, \cdots, \tau_n\}$, the second group consists of $\{\tau_n, \cdots, \tau_{2n-1}\}$, the next group consists of $\{\tau_{2n-1}, \cdots, \tau_{3n-2}\}$, etc.: that is, each group has $n$ elements and the first element in each group is the last element of the previous group.

The tuples corresponding to the $n$ elements $\{\tau_1, \cdots, \tau_n\}$ in the first group are given by:

$$\tau_j = (j, n{+}j, \ldots, (r{-}1)n{+}j) \quad \text{for} \quad 1 \leq j \leq n\,.$$

In particular, we have:

$$\tau_n = (n, 2n, \cdots, rn)\,.$$

More generally, for $2 \leq i \leq r$, let us denote the $i$-th group (of $n$ elements) by

$$\varphi_{ij} := \tau_{(i-1)n-(i-j)+1}\,, \quad 1 \leq j \leq n\,.$$

For $k<l$, let $[k,l] := (k, k{+}1, \cdots, l)$. Then for $1 \le j \le n$, we have,

$\varphi_{ij} = (\,(i{-}1)n{+}1{+}j{-}i,\ i\,n{+}1{+}j{-}i, \ldots, (r{-}1)n{+}1{+}j{-}i,\ [rn{+}2{-}i, rn]\,)\,.$

Here the succesive differences between the first $(r{+}1{-}i)$ entries is $n$, and the remaining entries are consecutive integers.

**Lemma 4.4.1.** (1) $\#\{\text{ admissible } S \text{ on } \tau_1\,\} = n^r$
(2) *For* $1 \le j \le n$, $\#\{\text{ admissible } S \text{ on } \tau_j\,\} = n^{r-1}(n{+}1{-}j)$.

*Proof.* (1) Let $S := (s_1, \cdots, s_r)$ be a typical admissible tuple on $\tau_1 = w_s$. The expression for $w_s$ as a $r$-tuple together with the admissibility of $S$ implies that there are $n$ ways of choosing $s_r$ from $\{(r-1)n + 1, (r-1)n+2, \cdots, rn\}$; having chosen $s_r$, there are $n$ ways of choosing $s_{r-1}$ from $\{s_r - n, \cdots, s_r - 1\}$, and so on.
(2) The proof is similar: at the first step, we have only $(n+1-j)$-choices for $s_r$ (since $s_r$ has to be $\ge (r{-}1)n{+}j$, there are $(n{+}1{-}j)$ ways of choosing $s_r$ from $\{(r-1)n+j, \cdots, rn\}$). The rest is as before. $\square$

More generally, we have:

**Lemma 4.4.2.** (1) $\#\{\text{ admissible } S \text{ on } \varphi_{i1}\,\} = n^{r+1-i}$.
(2) $\#\{\text{ admissible } S \text{ on } \varphi_{ij}\,\} = n^{r-i}(n{-}j{+}1)$.

*Proof.* (1) Since the last $(i-1)$ entries in $\varphi_{i1}$ are comprised of $[rn + 2 - i, rn]$, we have that in any admissible $S$ on $X(\varphi_{i1})$, the last $(i-1)$ entries are again comprised of $[rn + 2 - i, rn]$. With $s_{r+1-i}$, we have $n$ choices (note that $s_{r+1-i}$ may be chosen to be any entry from $[(r-1)n + 2 - i, rn + 1 - i]$). Rest of the proof is as in Lemma 4.4.1
(2) Again there is precisely one choice for $s_{r+2-i}, \cdots, s_r$. With $s_{r+1-i}$, we have $n + 1 - j$ choices (note that $s_{r+1-i}$ may be chosen to be any entry from $[(r-1)n + 1 + j - i, rn + 1 - i]$). The rest of the proof is as in Lemma 4.4.1 $\square$

**Pivots.** Recall that $1,\ n{+}1,\ 2n{+}1, \cdots, (r{-}2)n{+}1,\ (r{-}1)n{+}1$, the entries of $w_s$, are called the *pivots* of $w_s$.

For $1 \le i \le r(n{-}1){+}1$, let $\mathcal{A}_i$ denote the set of all admissible $S := (s_1, \cdots, s_r)$ on $\tau_i$, so that $\mathcal{A}_i \subset \mathcal{A}_{i-1}$. We shall denote $\mathcal{A}_1$ also by just $\mathcal{A}$. If $S = (s_1, \cdots, s_r) \in \mathcal{A}$ correponds to a Weyl group element $\tau$, then we shall denote $s_k$ also by $\tau(k)$. Let

$$Z = \{S \in \mathcal{A} \mid s_r = (r{-}1)n{+}1 = \tau_1(r)\}$$

**Remark 4.4.3.** (1) $\#\,Z = n^{r-1}$
(2) Let $S \in \mathcal{A}$. Then $p_S|_{X(\tau_2)}$ is non-zero if and only if $S \notin Z$. This is clear since:

$$\tau_2 = (2,\ n{+}2,\ 2n{+}2, \ldots, (r{-}2)n{+}2,\ (r{-}1)n{+}2)\,,$$

and hence $p_S|_{X(\tau_2)}$ is non-zero if and only if $s_r \geq (r-1)n+2$. Note that $\tau_2$ is the smallest (under $\geq$) in $\mathcal{A}$ such that $s_r \geq (r{-}1)n{+}2$.

(3) $\mathcal{A} = Z \,\dot\cup\, \mathcal{A}_2$

For $0 \leq j \leq r-1$, set

$$Z_j := \left\{ S = (s_1, \cdots, s_r) \in Z \;\middle|\; \begin{array}{c} s_i = (i{-}1)n{+}1 = \tau_1(i) \\ \text{for } j < i \leq r\,, \\ \text{and } s_j > (j{-}1)n{+}1 \end{array} \right\}$$

i.e., for $S \in Z_j$, $j$ is the largest such that $s_j$ is not a pivot. Note that:

(1) $Z = \dot\bigcup_{0\leq j\leq r-1} Z_j$.

(2) $Z_0 = \{w_s\}$

Let us arrange the elements of $Z$ lexicographically as $\{S_1 < S_2 < \cdots\}$, where

$$S_1 = (1,\, n{+}1,\, 2n{+}1, \ldots, (r{-}2)n{+}1,\, (r{-}1)n{+}1) = \tau_1\,.$$

$$S_2 = (2,\, n{+}1,\, 2n{+}1, \ldots, (r{-}2)n{+}1,\, (r{-}1)n{+}1)\,,$$

etc,. The element $S_2$ will play a crucial role in the discussion below.

4.5. **Degree two straightening relations.** In this subsection, we prove certain degree-two straightening relations among admissible Plücker coordinates on $X(w_s)$ which will be used for proving the linear independence of admissible Plücker coordinates on $X(w_s)$ in the next subsection.

We shall work with the open cell $X^\circ(w_s)$ in $X(w_s)$ using the notation of §4.2, particulary the normalized $rn \times r$ matrix presentation $M$ of a space $V \in X^\circ(w_s)$. We shall now compute $f_S := p_S|_{X^\circ(w_s)}$, for $S \in \mathcal{A}$ in terms of the entries in $M$. Further, if $S$ corresponds to a Weyl group element $\tau$, we shall denote $f_S$ also by $f_\tau$.

**Lemma 4.5.1.** *We have: (1)* $f_{\tau_1} = 1$ *; (2)* $f_{\tau_2} = a_{21}^r$ *; (3)* $f_{S_2} = a_{21}$

*Proof.* Clear by direct computations with the matrix $M$. □

**Lemma 4.5.2.** *Let* $S = (s_1, \cdots, s_r)$ *be in* $Z_j$*. Let* $S'$ *be obtained from* $S$ *by replacing* $s_{j+1}$ *by* $s_{j+1}+1$*.*

(1) $S'$ *is admissible; further,* $S'$ *is in* $Z_{j+1}$ *or* $\mathcal{A}_2$ *according as* $j < r{-}1$ *or* $j = r{-}1$*.*

(2) $f_{S_2} f_S = f_{\tau_1} f_{S'}$

*Proof.* (1) The admissibility of $S'$ follows from the facts that $s_j$ is not a pivot and $s_{j+1}$ is a pivot (since $S \in Z_j$); the latter assertion in (1) is clear from this.

(2) We compare the evaluations of $f_S, f_{S'}$ on the matrix $M$. We have $f_S(M)$ is the determinant of the matrix

$$\begin{pmatrix} M_j & 0_{j,\,r-j} \\ * & D_{r-j,\,r-j} \end{pmatrix}$$

where $M_j$ is the $j \times j$ sub matrix of $M$ with row indices $s_1, s_2, \cdots, s_j$ and column indices $1, 2, \cdots, j$, and $D_{r-j,\,r-j}$ is the diagonal $(r-j) \times (r-j)$ matrix $\operatorname{diag}(x, x, \cdots, x)$. Hence

$$(*) \qquad f_S(M) = x^{r-j}\Delta,$$

where $\Delta := \det M_j$.

Similarly, $f_{S'}(M)$ is the determinant of the matrix

$$\begin{pmatrix} M'_j & 0_{j+1,\,r-j-1} \\ * & D_{r-j-1,\,r-j-1} \end{pmatrix}$$

where $M'_j$ is the $(j+1) \times (j+1)$ submatrix of $M$ with row indices $s_1, s_2, \cdots, s_j, s_{j+1}+1$ and column indices $1, 2, \cdots, j+1$, and $D_{r-j-1,\,r-j-1}$ is the diagonal $(r-j-1) \times (r-j-1)$ matrix $\operatorname{diag}(x, x, \cdots, x)$. Hence

$$(**) \qquad f_{S'}(M) = x^{r-j-1}\Delta',$$

where $\Delta' := \det M'_j$.

But now the fact that $S, S'$ differ just in the $(j+1)$-th entry implies that the first $j$ columns of $M'_j$ are obtained by adding $a_{nj+2,\,1}$, $a_{(n-1)j+2,\,1}, \cdots,$ $a_{n+2,\,1}$ respectively to the $j$ columns of $M_j$ (in view of $\tau$-stability of $w_s$) and the last column of $M'$ consists of zeroes except the last entry which is $a_{2,1}$. (Note that for $1 \le i \le r$, in the $i$-th column of $A$, the $((i-1)n+1)$-th entry is $a_{11} = 1$, and the $((i-1)n+2)$-th entry is $a_{21}$.)

We obtain $\Delta' = a_{21}\Delta$, so that $(**)$ implies $f_{S'}(M) = x^{r-j-1}a_{21}\Delta$. Thus:

$$x f_{S'}(M) = x^{r-j}a_{21}\Delta = a_{21} f_S(M)\,,$$

which, with Lemma 4.5.1, implies:

$$f_{S_2}(M) f_S(M) = a_{21} f_S(M) = f_{S'}(M) = f_{\tau_1}(M) f_{S'}(M)$$

From this (2) follows. □

More generally, we have similar quadratic relations among $\{f_S,\, S \in \mathcal{A}_i\}$ for all $1 \le i \le r(n-1)$ as given by the Lemma below. Let us first fix some notation. Fix a $\varphi := \tau_l$ for $1 \le l \le r(n-1)$. We can identify $\varphi$ with some $\varphi_{ik}$ for $1 \le i \le r$ , $1 \le k \le n$ ; in fact, we may suppose that $k < n$, since $\varphi_{in} = \varphi_{i+1,\,1}$ (since for $i = r$ we have $\varphi_{rn} = \mathrm{id}$).

Let us enumerate the elements of $\mathcal{A}_\varphi = \mathcal{A}_l$ as $\{R_1, R_2, \ldots\}$ so that $\varphi = R_1 < R_2 < \cdots$ . We have:

$$\varphi = \varphi_{ik} = ((i{-}1)n{+}1{+}k{-}i,\ in{+}1{+}k{-}i, \ldots, (r{-}1)n{+}1{+}k{-}i,\ [rn{+}2{-}i, rn])\,.$$

Write $(i-1)n+1+k-i = p_{ik}n + q_{ik}$ where $1 \le q_{ik} \le n$; for simplicity of notation, let us denote $p_{ik}, q_{ik}$ by just $p, q$ respectively. Then

$$\varphi = (pn+q, (p+1)n+q, \cdots, (r+p-i)n+q, [rn+2-i, rn])\,.$$

As with $\tau_1$, we shall do computations on the open cell $X^\circ(\varphi)$ in $X(\varphi)$. The $\mathbb{Z} \times \mathbb{Z}_+$ matrix presentation for the elements of $X^\circ(\varphi)$ has the following description: The "relevant part" of the matrix presentation of a a generic point in $X^\circ(\varphi)$, may be thought of as a lower triangular $(r-p)n \times r$ matrix $M := (a_{gh})$ composed of $r-p$ blocks $A_{p+1}, \cdots, A_r$ of $n \times r$ matrices, where all the columns of $A_{p+1}$ except the first column consist of zeroes and the first column has the form

$$\begin{pmatrix} 0 \\ \vdots \\ a_{pn+q,\,1} \\ \vdots \\ a_{pn+n,\,1} \end{pmatrix}$$

Further, we have (in view of $\tau$-stability of $\varphi$) that the 1$^{\text{st}}$ column of $A_{p+1}$ repeats as the 2$^{\text{nd}}$ column of $A_{p+2}$, repeats as the 3$^{\text{rd}}$ column of $A_{p+3}$, and so on. Similarly, the 1$^{\text{st}}$ column of $A_{p+2}$ repeats as the 2$^{\text{nd}}$ column of $A_{p+3}$, repeats as the 3$^{\text{rd}}$ column of $A_{p+4}$, and so on.

Further, we define the pivotal rows to be those with indices $\{mn{+}q \mid p \le m \le r{+}p{-}i\}$, together with all the entries of $[rn{+}2{-}i, rn]$. In these rows, the matrix $M$ has a single entry 1 and all other entries 0. That is:

$$\begin{aligned} a_{mn+q,\, m+1-p} &= 1, \quad p \le m \le r{+}p{-}i \\ a_{rn+s-i,\, r+s-i} &= 1, \quad 2 \le s \le i \\ a_{(mn+q)\, j} &= 0, \quad j \ne m{+}1{-}p\,. \end{aligned}$$

entry, $2 \le s \le i$.) In the discussion below, $f_S$ will denote $p_S\,|_{\,X^\circ(\varphi)}$. Let $\varphi'$ denote $\varphi_{i\,k+1}$. Note that $X(\varphi')$ is a divisor in $X(\varphi)$.

**Lemma 4.5.3.** *(1)* $f_\varphi = 1$ *; (2)* $f_{\varphi'} = a^{r+1-i}_{pn+q+1,\,1}$ *; (3)* $f_{R_2} = a_{pn+q+1,\,1}$ .

The proof is similar to that of Lemma 4.5.1.

**Remark 4.5.4.** Let

$$Z(\varphi) = \{S \in \mathcal{A}_\varphi \mid s_{r+1-i} = \varphi(r{+}1{-}i)\}\,.$$

Then we have:

1. $\# Z(\varphi) = n^{r-i}$

2. Let $S \in \mathcal{A}_\varphi$. Then $p_S|_{X(\varphi')}$ is non-zero if and only if $S \notin Z(\varphi)$. This is clear because $\varphi'(r+1-i) = \varphi(r+1-i) + 1$, and hence $p_S|_{X(\varphi')}$ is non-zero if and only if $s_{r+1-i} \geq \varphi(r+1-i) + 1$. Indeed, $\varphi'$ is the smallest (under $\geq$) in $\mathcal{A}_\varphi$ such that $s_{r+1-i} \geq \varphi(r+1-i)+1$.

3. $\mathcal{A}_\varphi = Z(\varphi) \dot\cup \mathcal{A}_{\varphi'}$ .

For $0 \leq j \leq r-i$, set:

$$Z_j(\varphi) := \left\{ S = (s_1, \cdots, s_r) \in Z(\varphi) \;\middle|\; \begin{array}{c} s_m = \varphi(m) \\ \text{for } j < m \leq r+1-i, \\ \text{and } s_j > \varphi(j) \end{array} \right\},$$

i.e., for $S \in Z_j(\varphi)$, $j$ is the largest such that $s_j > \varphi(j)$. Note:

1. $Z(\varphi) = \dot\bigcup_{0 \leq j \leq r-i} Z_j(\varphi)$.

2. $Z_0 = \{\varphi\}$

**Lemma 4.5.5.** *Let $S = (s_1, \cdots, s_r)$ be in $Z_j(\varphi)$. Let $S'$ be obtained from $S$ by replacing $s_{j+1}$ by $s_{j+1}+1$.*

(1) *$S'$ is admissible; further, $S'$ is in $Z_{j+1}(\varphi)$ or $\mathcal{A}_{\varphi'}$ according as $j < r-i$ or $j = r-i$.*

(2) $f_{R_2} f_S = f_\varphi f_{S'}$

The proof is similar to that of Lemma 4.5.2.

4.6. **Linear independence of admissible Plücker coordinates.** To show the independence of $\{p_S \mid S \in \mathcal{A}_\varphi\}$, we first prove the independence of $\{p_S \mid S \in Z_j(\varphi)\}$ for $0 \leq j \leq r-i$. We use induction on $\dim X(\varphi)$, the starting point being $\varphi = \mathrm{id}$, for which the result is clear.

Let then $\dim X(\varphi) > 0$.

First, let $j < r-i$, and suppose

$$(*) \qquad \sum_{S \in Z_j(\varphi)} a_S f_S = 0, \quad a_S \in K\,.$$

Multiplying the relation by $f_{R_2}$, by Lemma 4.5.5 it reduces to:

$$f_\varphi \sum_{S' \in Z_{j+1}(\varphi)} a_S f_{S'} = 0\,.$$

$S'$ being as in Lemma 4.5.5. Hence cancelling $f_\varphi$, it reduces to

$$\sum_{S' \in Z_{j+1}(\varphi)} a_S f_{S'} = 0\,.$$

Hence by decreasing induction on $j$, we obtain $a_S = 0$, for all $S$.

Let now $j = r - i$. Multiplying the relation $(*)$ by $f_{R_2}$, it reduces to (in view of Lemma 4.5.5):

$$f_\varphi \sum_{S' \in \mathcal{A}_{\varphi'}} a_S f_{S'} = 0 \, .$$

Cancelling $f_\varphi$, and restricting to $X(\varphi')$, we obtain by induction on dim $X(\varphi)$ that $a_S = 0$ for all $S$.

Next, we prove the linear independence of $\{p_S \mid S \in Z(\varphi)\}$. Suppose:

$$(**) \qquad \sum_{S \in Z(\varphi)} a_S f_S = 0, \quad a_S \in K \, .$$

As above, multiplying the relation (**) by $f_{R_2}$, cancelling $f_\varphi$ and restricting to $X(\varphi')$, we obtain $a_S = 0$, for all $S \in Z_{r-i}(\varphi)$. Hence (**) reduces to:

$$(***) \qquad \sum_{\substack{S \in Z_j(\varphi) \\ j < r-i}} a_S f_S = 0, \quad a_S \in K \, .$$

Again multiplying the relation (***) by $f_{R_2}$, cancelling $f_\varphi$ and using the first step above, we obtain $a_S = 0$, for all $S \in Z_{r-i-1}(\varphi)$. (Note that in the resulting relation, $\{a_S \mid S \in Z_{r-i-1}(\varphi)\}$ occur as coefficients of the corresponding $S' \in Z_{r-i}(\varphi)$.) Thus proceeding, we obtain $a_S = 0$ for all $S \in Z(\varphi)$ appearing in (**). Thus we obtain:

**Proposition 4.6.1.** *$Z(\varphi)$ is linearly independent.*

We next prove the linear independence of $\{p_S \mid S \in \mathcal{A}_\varphi\}$. Since $\mathcal{A}_\varphi = Z(\varphi) \dot\cup \mathcal{A}_{\varphi'}$, we may write a linear relation as:

$$(\dagger) \qquad \sum_{S \in Z(\varphi)} a_S f_S + \sum_{S \in \mathcal{A}_{\varphi'}} b_S f_S = 0 \, .$$

Restricting (†) to $X(\varphi')$, we first conclude that $b_S = 0$, for all $S \in \mathcal{A}_{\varphi'}$. Then (†) reduces to:

$$\sum_{S \in Z(\varphi)} a_S f_S = 0 \, .$$

In view of Proposition 4.6.1, it follows that $a_S = 0$, for all $S \in Z(\varphi)$. Thus we obtain:

**Proposition 4.6.2.** *$\{p_S \mid S \in \mathcal{A}_{\tau_\ell}\}$ is linearly independent for all $1 \le \ell \le (r-1)n+1$.*

Proposition 4.6.2 together with the generation by admissible $p_S$ (Proposition 4.3.4) implies:

**Theorem 4.6.3.** *$\{p_S, S \in \mathcal{A}_{\tau_\ell}\}$ is a basis for $H^0(X(\tau_\ell), L_0)$ for all $1 \leq \ell \leq (r-1)n+1$.*

As an immediate Corollary, we obtain:

**Corollary 4.6.4.** *The straightening relations on $X(\tau_l)$ as given by Proposition 4.3.4 give a set of generators for the the degree one part of the ideal of $X(\tau_l)$ considered as a closed subvariety of $Gr(sn, V_s)$.*

As another consequence, we have:

**Theorem 4.6.5.** *The shuffle relations among the Plücker co-ordinates (of Corollary 4.3.3) give a set of generators for the degree one part of the ideal defining the affine Grassmannian inside the infinite Grassmannian.*

*Proof.* As seen in §3 (Proposition 3.1, Remark3.2), we have that the affine Grassmannian is the inductive limit of the $X(w_s)$, and the result follows from Theorem 4.6.3, and Corollary 4.6.4. $\square$

**Conjecture:** The affine Grassmannian is cut out inside the infinite Grassmannian by the (linear) shuffle relations.

## 5. Application to Nilpotent Orbit Closures

5.1. **$\mathcal{P}$-stable Affine Schubert Varieties.** Define $W_{st}^P = \{w = (c_1, \ldots, c_n) \in W \mid c_1 + \cdots + c_n = 0, c_1 \leq \cdots \leq c_n\} \subset W^P$. One can check that for $w \in W_{st}^P$, $X(w)$ is stable by left translations by $\mathcal{P}$ (and not just by $\mathcal{B}$), and thus by $\mathrm{SL}_n(K) \subset \mathcal{P}$. Let $w \in W_{st}^P$. There exists a $s$ such that $X(w) \subset X(w_s)$ (in fact there are infinitely many such $s$); we consider $X(w)$ inside some fixed $X(w_s)$.

Define $\lambda_w = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ by $\lambda_i = -c_i + s$, $1 \leq i \leq n$. Note that $nc_n \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0$ ($\lambda_n \geq 0$ is implied by $X(w) \subset X(w_s)$) and $\lambda_1 + \cdots + \lambda_n = ns$. Thus $\lambda_w$ is a partition of $ns$ with at most $n$ non-zero rows. Conversely, for any partition $\lambda$ of $ns$ with at most $n$ rows, we have that $\lambda = \lambda_w$ for a unique $w \in W_{st}^P$, which we refer to as $w_\lambda$ (we will often write $X(\lambda)$ to refer to $X(w_\lambda)$). Thus there is a bijection between $\mathcal{P}$ stable affine Schubert varieties in $X(w_s)$ and partitions of $ns$ with at most $n$ rows, i.e., column lengths are $\leq n$.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\mu = (\mu_1, \ldots, \mu_m)$ both be partitions of $m \in \mathbb{N}$. We say that $\lambda \geq \mu$, if $\sum\limits_{i=1}^{j} \lambda_i \geq \sum\limits_{i=1}^{j} \mu_i$ for $j = 1, \ldots, m$. This order is referred to as the *dominance* order. The following Lemma can be easily verified.

**Lemma 5.1.1.** *Let $\lambda$, $\mu$ be partitions of $sn$ with at most $n$ rows. Then $w_\lambda \geq w_\mu \iff \lambda \geq \mu$ in the dominance order.*

For $\mu$ a partition of $sn$ with at most $n$ rows, let $\lambda$ be the conjugate partition. Let $E = \mathbb{C}^n$. The following result is shown in [25, 21].

**Theorem 5.1.2.** *$V^\star_{d\Lambda_0}(w_\lambda) \cong S_{\lambda_1^m}E \otimes \ldots \otimes S_{\lambda_s^m}E$ as $SL(E)$ modules (here, for a dominant weight $\nu = (\nu_1, \cdots, \nu_s)$, $L_\nu E$ denotes the associated Schur module).*

Applying Theorem 3.3, we obtain the following:

**Corollary 5.1.3.** *$K[X(\mu)]_d \cong L_{\lambda_1^m}E \otimes \ldots \otimes L_{\lambda_s^m}E$ as $SL(E)$ modules.*

5.2. **Nilpotent orbit closures.** In the rest of this section, we discuss possible applications of our approach to nilpotent orbit closures in positive characteristics.

Let $\mathcal{N}$ denote the set of all nilpotent matrices in $M_{n\times n}(K)$; it is a closed affine subvariety of $M_{n\times n}(K)$. The group $\mathrm{GL}_n(K)$ acts on $\mathcal{N}$ by conjugation. Each orbit contains precisely one matrix in Jordan canonical form (up to order of the Jordan blocks). Thus the orbits are indexed by partitions $\mu$ of $n$, i.e. $\mu = (\mu_1, \ldots, \mu_n)$, $n \geq \mu_1 \geq \cdots \geq \mu_n \geq 0$, $\mu_1 + \cdots + \mu_n = n$. The orbit corresponding to the partition $\mu = (\mu_1, \ldots, \mu_n)$ will be denoted by $\mathcal{N}^0_\mu$, and its closure by $\mathcal{N}_\mu$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, the conjugate partition of $\mu$. Then

$$\mathcal{N}^0_\mu = \left\{ N \in M_{n\times n}(K) \mid \operatorname{rank}(N^i) = n - \sum_{j=1}^{i} \lambda_j, i = 1, \ldots, n \right\},$$

$$\mathcal{N}_\mu = \left\{ N \in M_{n\times n}(K) \mid \operatorname{rank}(N^i) \leq n - \sum_{j=1}^{i} \lambda_j, i = 1, \ldots, n \right\}.$$

These two facts imply:

**Proposition 5.2.1.** *$\mathcal{N}^0_{\mu'} \subset \mathcal{N}_\mu \iff \mu' \leq \mu$ in the dominance order.*

In particular, $\mathcal{N}_{(n)}$ contains all nilpotent orbits; thus it equals $\mathcal{N}$.

Next we describe an isomorphism due to Lusztig between the nilpotent orbit closures and open subsets of certain affine Schubert varieties. Let $\kappa = w_s$ with $s = 1$, so that $r = n$; thus, with the convention in §4.2, $\kappa$ corresponds to the $n$-tuple $(1, n+1, 2n+1, \ldots, n(n-1)+1)$. A generic point of $X(\kappa)$ has a $\mathbb{Z}\times\mathbb{Z}_+$ matrix presentation with the relevant part $M$ being a $n^2 \times n$ matrix; further, $M$ consists of $n$ blocks of $n \times n$ matrices:

$$M = \begin{pmatrix} A_1 \\ \vdots \\ A_{n-1} \\ A_n \end{pmatrix}.$$

Define $X'(\kappa) = \{M \in X(\kappa) \mid p_L(M) \neq 0\}$, where $L$ is the $n$-tuple $(n(n-1)+1, \dots, n^2)$ (note that $p_L(M) = |A_n|$). Then $X'(\kappa)$ embeds into the affine subspace of $M_{n^2\times n}(K)$ with lowest block equal to the identity. For $\lambda$ a partition of $n$, define $X'(\lambda) = X(\lambda) \cap X'(\kappa)$ (here, $X(\lambda) = X(w_\lambda)$).

Consider the morphism $\phi : \mathcal{N} \to M_{n^2\times n}$ given by

$$\phi(N) = \begin{pmatrix} N^{n-1} \\ \vdots \\ N \\ I \end{pmatrix}.$$

Lusztig has shown (cf. [17], [18], [20])

**Theorem 5.2.2.** *$\phi|_{\mathcal{N}_\mu} : \mathcal{N}_\mu \to X'(\mu)$ is an isomorphism of affine varieties.*

For the rest of this section, we shall denote

$$E = K^n, \quad Y = M_n(K)\,, \quad A = K[x_{i,j}]_{1\le i,j\le n}\,,$$

$K$ being our algebraically closed field of arbitrary characteristic. We further denote $X := (x_{i,j})$ the $n \times n$ generic matrix, $J_\mu =$ the defining ideal of $\mathcal{N}_\mu$, and $A_\mu := A/J_\mu$ the coordinate ring of $\mathcal{N}_\mu$.

We are interested in the properties of the coordinate ring $A_\mu$ such as normality, rational singularities, equations, etc. Since normality and rational singularities are known, we really want to concentrate on the equations.

By the results of [22] on the simultaneous Frobenius splitting, the coordinate ring $A_\mu$ is normal and Cohen-Macaulay (in fact the Springer type resolution of $\mathcal{N}_\mu$ is rational). Donkin (cf. [3]) showed that the coordinate ring of $\mathcal{N}$ has a good filtration i.e. it is filtered by the Schur functors.

The defining equations are known (in a characteristic free way) for the following two classes of partitions:

(a) $\mu = (r, 1^{n-r})$ is a hook. These are nilpotent matrices of rank $\le r-1$. The defining ideal is generated by the $r$ minors of a generic $n \times n$ matrix and the invariants (namely, the coefficients of the characteristic polynomial). These varieties are complete intersections in the corresponding determinantal varieties.

(b) $\mu = (2^r, 1^{n-2r})$. These are the nilpotent matrices with the square being zero, of rank $\le r$. In this case the defining equations are the entries of the square of our matrix, the invariants and the $(r+1)$ minors. The minimal generators are the entries of the square of the matrix and one irreducible representation of highest weight $(1^{r+1}, 0^{n-2r-2}, (-1)^{r+1})$ given by cosets of $(r+1)$ minors.

Let $X$ be a generic $n \times n$ matrix. Consider the following $n^2 \times n$ matrix $M$:

$$M := M(X) := \begin{pmatrix} X^{n-1} \\ \vdots \\ X \\ I \end{pmatrix}$$

For a given $\mu$ we denote by $\mathcal{F}_{m,\mu}$ the span in $A_\mu$ of the products of $\leq m$ cosets of maximal minors of the matrix $M$ in $A_\mu$. We have by definition

$$\mathcal{F}_{m,\mu} \subset \mathcal{F}_{m+1,\mu}$$

We denote by $M(i_1, \ldots, i_n)$ the maximal minor of $M$ corresponding to the rows $i_1, \ldots, i_n$, for $1 \leq i_1 < \ldots < i_n \leq n^2$.

Let us start by formulating a general conjecture regarding the spaces $\mathcal{F}_{m,\mu}$. Let $E = K^n$.

**Conjecture:** Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be the partition conjugate to $\mu$. Then there is a characteristic free isomorphism of $SL(E)$-modules

$$\mathcal{F}_{m,\mu} = L_{\lambda_1^m} E \otimes \ldots \otimes L_{\lambda_s^m} E$$

where for a partition $\nu = (\nu_1, \ldots, \nu_r), L_\nu E$ denotes the Weyl module with highest weight $(\nu_1, \ldots, \nu_r)$.

**Remark 5.2.3.** In characteristic 0, the conjecture follows from Corollary 5.1.3 (cf. [21, 25]).

Our goal is to provide the explicit straightening of the products of minors of $M$ to the "standard products" of maximal minors. Such straightening gives another presentation of the ring $A_\mu$ with the set of generators consisting of minors spanning $\mathcal{F}_{1,\mu}$ other than the minor $M(n(n-1)+1, n(n-1)+2, \ldots, n^2)$ which is equal to 1. Notice that these generators include the entries $x_{i,j} = \pm M(n(n-2)+i, n(n-1)+1 \ldots, n(n-1)+j-1, n(n-1)+j+1, \ldots, n^2)$ for $1 \leq i, j \leq n$. In view of Theorem 5.2.2 we have:

**Proposition 5.2.4.** *The straightening relations on $X(\mu)$ among the Plćker co-ordinates(with $M(n(n-1)+1, n(n-1)+2, \ldots, n^2)$ specialized to $1$) give a set of generators of the defining ideal $J_\mu$ (here, $X(\mu)$ is the affine Schubert variety associated to $\mu$).*

The trouble is that even if such a straightening is described, still in order to get a set of generators of $J_\mu$ (in terms of the matrix entries

$x_{ij}$), we need to replace the generators of $\mathcal{F}_{1,\mu}$ other than the minors $M(n(n-2)+i, n(n-1)+1\ldots, n(n-1)+j-1, n(n-1)+j+1, \ldots, n^2)(= x_{ij})$ for $1 \leq i, j \leq n$, by suitable polynomial expressions in the $x_{ij}$'s.

5.3. **Kostka-Foulkes polynomials.** We will describe the equations of the ideals $J_\mu$ over the field of characteristic zero. This description was given in [27]. Here we give a proof that brings out the role of the spaces $\mathcal{F}_{1,\mu}$.

Let us fix $n$ and $\mu$. Let us also fix a dominant integral weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ for $GL(n)$, namely, $\alpha_i \in \mathbb{Z}$ and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. We denote by $S_\alpha E$, the Schur module associated to $\alpha$. The ring $A_\mu$ is a graded ring. Let us denote by $m_{\alpha,\mu,i}$ the multiplicity of $S_\alpha E$ in the $i$-th graded component of $A_\mu$. We define the series

$$P_{\alpha,\mu}(q) := \sum_{i\geq 0} m_{\alpha,\mu,i}\ q^i$$

We call the series $P_{\alpha,\mu}(q)$ the Poincare series corresponding to the weight $\alpha$ and to the orbit $\mathcal{N}_\mu^0$. Notice that the series $P_{\alpha,\mu}(q)$ is non zero only when $\alpha_1 + \ldots + \alpha_n = 0$. We will assume this throughout this section. All series $P_{\alpha,\mu}(q)$ are polynomials since by a formula of Kostant the total multiplicity of $S_\alpha E$ in $A_{(n)}$ equals the multiplicity of the zero weight in $S_\alpha E$.

The polynomials $P_{\alpha,\mu}(q)$ have a very interesting connection with the Kostka-Foulkes polynomials (see [19] for the definition of Kostka-Foulkes polynomials). This connection is related to Kraft's construction of intersecting with the diagonal which we now describe.

Let $I \subset A(= K[x_{ij}])$ be the defining ideal of the set of diagonal matrices, i.e. the ideal generated by the elements $x_{i,j}$ with $i \neq j$. We define the algebras

$$B = A/I, B_{\mu'} := A/(I + J_\mu)$$

where $\mu'$ is the partition conjugate to $\mu$; the reason for using the conjugate partition $\mu'$ to label the algebra is that socle of $B_{\mu'}$ is the Specht module $\Sigma^{\mu'}$).

The algebras $B, B_{\mu'}$ are graded algebras. There are natural actions of the symmetric group $S_n$ on the graded algebras $B, B_{\mu'}$. Indeed, we can embed $S_n$ into $GL(E)$ by sending a permutation to the corresponding permutation matrix. This defines an action of $S_n$ on $A$. The ideals $I, I + J_\mu$ are $S_n$-stable; hence we obtain natural actions of $S_n$ on $B, B_{\mu'}$. These actions are compatible with the natural surjections $B \to B_{\mu'}$, $B_\mu \to B_\nu$ (where $\nu$ is less than $\mu$ in the dominance order). We denote by $I_{\mu'}$ the kernel of the surjection $B \to B_\mu$, i.e., $B_{\mu'} = B/I_{\mu'}$.

We have natural restriction maps $\psi : A \to B$, $\psi_\mu : A_\mu \to B_{\mu'}$. We will denote the image of the element $x_{i,i}$ in the algebras $B, B_{\mu'}$ by $y_i$. The action of $S_n$ on $B, B_\mu$ is given by

$$\sigma(y_i) = y_{\sigma(i)}$$

For a partition $\lambda$ of $n$ we denote by $\Sigma^\lambda$ the Specht module. Let $R$ be the representation ring of the symmetric group $S_n$. For a representation $V$ of $S_n$ we denote by $[V]$ its class in $R$. We also consider the polynomial ring $R[q]$ over $R$ in a variable $q$. For a partition $\mu$ of $n$ we define the element $\tilde{K}_\mu(q) \in R[q]$ (cf. [19]) as

$$\tilde{K}_\mu(q) = \sum_{i \geq 0} [B_{\mu,i}] q^i$$

where $B_{\mu,i}$ denotes the $i$-th graded component of $B_\mu$. We can write:

$$\tilde{K}_\mu(q) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q)[\Sigma^\lambda]\,.$$

The polynomials $\tilde{K}_{\lambda,\mu}(q)$ were extensively studied in several contexts. In [19], ch III. §6] they were defined as the transition polynomials for the Hall-Littlewood symmetric functions. In [2, 4] it was shown that the definition above and the one in [19] agree. For two partitions $\lambda, \mu$ denote

$a_{\mu,\lambda,i} :=$ the multiplicity of $\Sigma^\lambda$ in the $i^{th}$ graded component of $B_\mu$

Define

$$r_{\mu,\lambda}(q) = \sum_{i \geq 0} a_{\mu,\lambda,i} q^i$$

The following Proposition was conjectured in [11] and proved in [2].

**Proposition 5.3.1.** *Let $\mu$ be a partition of $n$ and let $\mu' := (\mu'_1, \dots, \mu'_t)$ be the conjugate partition.*

(1) *The representation of $S_n$ on $B_\mu$ is isomorphic to the induced representation from the trivial representation of $S_{\mu'_1} \times \ldots \times S_{\mu'_t}$ to $S_n$.*
(2) *The polynomial $r_{\mu,\lambda}(q)$ is equal to the polynomial $\tilde{K}_{\mu,\lambda}(q) := q^{-n(\lambda)} K_{\mu,\lambda}(q^{-1})$ where $K_{\mu,\lambda}(q)$ is the Kostka-Foulkes polynomial(cf. [19]).*

The families of polynomials $P_{\alpha,\mu}(q)$ and $\tilde{K}_{\lambda,\mu}(q)$ overlap in several important cases. The first occurrence is when $\mu = (n)$. Let $\mu$ be a partition of $n$ and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a dominant weight for $GL(n)$ with $\alpha_1 + \ldots + \alpha_n = 0$. Let $\alpha_n = -k$. We can define a partition $\lambda := \lambda(\alpha)$ of $nk$ by setting $\lambda = (\alpha_1 + k, \dots, \alpha_n + k)$.

**Proposition 5.3.2.** *(cf.* [5, 6]*) We have* $P_{\alpha,(n)}(q) = \tilde{K}_{\lambda,(k^n)}(q)$.

The next occurrence deals with the following special kind of dominant weights for $GL(E)$: let $\mathcal{W}_1^n$ denote the set of dominant weights $\alpha = (\alpha_1, \ldots, \alpha_n)$ for $GL(E)$ with $\alpha_1 + \ldots + \alpha_n = 0$, $\alpha_n = -1$. Such weights are sometimes called *the weights of level one*. Let $\mathcal{P}_n$ denote the partitions of $n$. We have a bijection

$$\diamond : \mathcal{W}_1^n \to \mathcal{P}_n,\ \diamond(\alpha) = (\alpha_1 + 1, \ldots, \alpha_n + 1)$$

**Theorem 5.3.3.** *Let* $\mu$ *be a partition of* $n$ *and let* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *be in* $\mathcal{W}_1^n$. *Let* $\lambda = \diamond(\alpha)$, *i.e.* $\lambda = (\alpha_1 + 1, \ldots, \alpha_n + 1)$. *Then we have*

$$P_{\alpha,\mu}(q) = \tilde{K}_{\lambda,\mu'}(q)$$

*Proof.* This follows from [28], §6. □

Theorem 5.3.3 gives another interpretation to the spaces $\mathcal{F}_{m,\mu}$ related to the matrix $M$ defined at the beginning of this section.

We now give a representation-theoretic interpretation for $\mathcal{F}_{1,\mu}$ in characteristic zero.

**Proposition 5.3.4.** *Let* $K$ *be a field of characteristic zero. Let* $\mathcal{G}_{m,\mu}$ *be the isotypic component of* $A_\mu$ *consisting of all representations* $S_{(\alpha_1,\ldots,\alpha_n)}E$ *with* $\alpha_n \geq -m$. *Then* $\mathcal{G}_{m,\mu} = \mathcal{F}_{m,\mu}$.

*Proof.* Let first $m = 1$. We know (by representation theory) that $\mathcal{F}_{1,\mu} \subset \mathcal{G}_{1,\mu}$ Indeed, the maximal minors of the matrix $M$ span the subspace whose $GL(n)$-representation type is a sub representation of $\bigwedge^n E^* \otimes W$ where $W$ is a polynomial representation of $GL(n)$ of degree $n$ (degree being with respect to the matrix entries).

We will show that for dimension reasons we in fact have equality. For a weight $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_n = -1$, $\alpha_1 + \ldots + \alpha_n = 0$, consider the partition $\nu(\alpha) := (\alpha_1 + 1, \ldots, \alpha_n + 1)(= \diamond(\alpha))$. Then by Theorem 5.3.3, the subspace $\mathcal{G}_{1,\mu}$ has the decomposition to isotypic components given by Kostka-Foulkes polynomials:

$$\mathcal{G}_{1,\mu} = \bigoplus_{\substack{\alpha\ :\ \alpha_n=-1 \\ \alpha_1+\cdots+\alpha_n=0}} \tilde{K}_{\mu',\nu(\alpha)}(1)\, S_\alpha E\,.$$

Thus we see that as an $SL(E)$-module, we have

$$\mathcal{G}_{1,\mu} = S_{\lambda_1}E \otimes \ldots \otimes S_{\lambda_s}E\,,$$

$(\lambda_1, \cdots, \lambda_s)$ being the partition conjugate to $\mu$; hence we obtain $\mathcal{G}_{1,\mu} = \mathcal{F}_{1,\mu}$. (Note that in view of Remark 5.2.3 and Corollary 5.1.3, $\mathcal{F}_{1,\mu} = L_{\lambda_1}E \otimes \ldots \otimes L_{\lambda_s}E$ .)

For arbitrary $m$ it is clear that the subspace $\mathcal{G}_{m,\mu}$ contains $\mathcal{F}_{m,\mu}$. The equality is also clear because the space $\mathcal{F}_{1,\mu}$ generates $A_\mu$ as an algebra, and the equality is true for the case of the nullcone $\mu = (n)$ (we have the dimension argument working for arbitrary $m$ in that case). □

**Corollary 5.3.5.** *Let $\mathcal{F}_{1,\mu,d}$ be the $d$-th graded component of the space $\mathcal{F}_{1,\mu}$. we have*

$$\sum_{d\geq 0} q^d \mathrm{char}(\mathcal{F}_{1,\mu,d}) = \bigoplus_{\substack{\alpha\ :\ \alpha_n=-1\\ \alpha_1+\cdots+\alpha_n=0}} \tilde{K}_{\mu',\nu(\alpha)}(q) S_\alpha E$$

5.4. **Equations of nilpotent orbit closures.** Let us investigate the setting of Theorem 5.3.3 more closely. Notice that the Specht module $\Sigma^{\diamond(\alpha)}$ can be defined as the set of vectors in $S_\alpha E$ of weight zero. This means that the restriction map $\psi_\mu : A_\mu \to B_{\mu'}$ takes the isotypic component of type $\alpha$ in $A_\mu$ to the isotypic component of type $\lambda := \diamond(\alpha)$ in $B_{\mu'}$. Thus Theorem 5.3.3 can be strengthened as:

**Theorem 5.4.1.** *Let $n, \lambda, \alpha, \mu$ be as above. Let $A^\alpha_\mu$, $B^\lambda_\mu$ denote the isotypic components of $A_\mu, B_\mu$ respectively. Then the restriction map induces isomorphisms*

$$\psi^\alpha_\mu : A^\alpha_\mu \to B^\lambda_{\mu'}$$

*Proof.* First we prove the proposition for $\mu = (n)$. Recall that $T_1, \ldots, T_n$ (the coefficients of the characteristic polynomial of the generic $n \times n$ matrix) are the basic invariants in $A$. Denote $e_j := e_j(y_1, \ldots, y_n)$ the elementary symmetric functions in $y_1, \ldots, y_n$. Consider the Koszul complexes $K(T_1, \ldots, T_n; A)$ and $K(e_1, \ldots, e_n; B)$. The restriction map induces a map of complexes

$$\mathrm{res} : K(T_1, \ldots, T_n; A)^\alpha \to K(e_1, \ldots, e_n; B)^\lambda$$

which is an epimorphism on each term. Both complexes are acyclic with zero homology groups being $A^\alpha_{(n)}$ and $B^\lambda_{(1^n)}$ respectively. It follows by diagram chase that the induced restriction map $\psi^\alpha_{(n)} : A^\alpha_{(n)} \to B^\lambda_{(1^n)}$ is an epimorphism. Since the dimensions of both spaces are the same by Theorem 5.3.3, the map is an isomorphism as claimed.

To prove the general case we consider the commutative diagram

$$\begin{array}{ccc} A^\alpha_{(n)} & \overset{\psi^\alpha_{(n)}}{\rightarrow} & B^\lambda_{(1^n)} \\ \downarrow & & \downarrow \\ A^\alpha_\mu & \overset{\psi^\alpha_\mu}{\rightarrow} & B^\lambda_{\mu'} \end{array}$$

Since the vertical maps and the upper horizontal map are epimorphisms it follows that $\psi^\alpha_\mu$ is also an epimorphism. Again by Theorem 5.3.3 the vector spaces $A^\alpha_\mu$ and $B^\lambda_{\mu'}$ have the same dimension, so the map $\psi^\alpha_\mu$ is an isomorphism and we are done. $\square$

Let $A^d_\mu, B^d_{\mu'}$ denote the elements of degree $d$ in $A_\mu, B_{\mu'}$ respectively. We will need the following lemma regarding the multiplication by elements of degree one in $A$ and $B$.

**Lemma 5.4.2.** *Let $\mu, \alpha, \lambda$, be as above. Consider a representation $S_\alpha E$ contained in $A^d_\mu$ and its restriction $\Sigma^\lambda$ contained in $B^d_{\mu'}$. Then the vector subspace $(\Sigma^\lambda)B^1_{\mu'}$ of $B^{d+1}_{\mu'}$ is the epimorphic image (by the restriction maps in corresponding weights) of the vector subspace $(S_\alpha E)A^1_\mu$ of $A^{d+1}_\mu$.*

*Proof.* The vector space $B^1_{\mu'}$ is isomorphic to the Specht module $\Sigma^{(n-1,1)}$ (except for the trivial case $\mu = (1^n)$). Let us investigate the tensor product $\Sigma^\lambda \otimes_K \Sigma^{(n-1,1)}$ as a $S_n$-module with the diagonal action of $S_n$. Let us denote by $[\lambda, 1]$ the set of partitions which differ from $\lambda$ by exactly one box. Let $d(\lambda)$ denote the number of corner boxes in $\lambda$ diminished by one. It is well known that the above tensor product has the following decomposition.

$$\Sigma^\lambda \otimes_K \Sigma^{(n-1,1)} = \oplus_{\nu\in[\lambda,1]}\Sigma^\nu \oplus (\Sigma^\lambda)^{d(\lambda)}.$$

The vector space $A^1_\mu$ is isomorphic to $S_{(1,0^{n-2},-1)}E$ (except for the trivial case $\mu = (1^n)$). Littlewood-Richardson Rule implies the tensor product decomposition

$$S_\alpha E\otimes S_{(1,0^{n-2},-1)}E = \oplus_{\nu\in[\lambda,1]}S_{\diamond^{-1}(\nu)}E\oplus(S_\alpha E)^{d(\lambda)}\oplus\oplus_{\beta\notin\mathcal{P}(1)}S_\beta E^{\oplus m(\alpha,\beta)}\,.$$

Here $m(\alpha, \beta)$ denotes the multiplicity of $S_\beta E$.

Moreover, since the Specht module $\Sigma^{\diamond(\alpha)}$ is obtained from $S_\alpha E$ by restricting to weight 0, one concludes that the restriction to weight zero induces the epimorphism from the first two summands in the second decomposition to the tensor product $\Sigma^\lambda \otimes_W \Sigma^{(n-1,1)}$. Let us call the sum of the first two summands in the second decomposition by $(S_\alpha E\otimes S_{(1,0^{n-2},-1)}E)_{\mathcal{P}(1)}$. The lemma follows by a chase of the diagram:

$$\begin{tikzcd}
(S_\alpha E \otimes S_{(1,0^{n-2},-1)}E)_{\mathcal{P}(1)} \arrow[r, hook] \arrow[dr, "h"'] & S_\alpha E \otimes S_{(1,0^{n-2},-1)}E \arrow[r] \arrow[d] & A^1_\mu \otimes A^d_\mu \arrow[r] \arrow[d] & A^{d+1}_\mu \arrow[d] \\
 & \Sigma^{(n-1,1)} \otimes \Sigma^\lambda \arrow[r] & B^1_{\mu'} \otimes B^d_{\mu'} \arrow[r] & B^{d+1}_{\mu'}
\end{tikzcd}$$

taking into account that the map $h$ and all vertical maps are epimorphisms. $\square$

Theorem 5.4.1 and Lemma 5.4.2 allow us to describe explicitly the generators of the ideals $J_\mu$, thus giving a new proof of the result of the last author (cf. [27]). The main idea is that we know for other reasons that the equations of $J_\mu$ have to belong to the subspace $\mathcal{F}_{1,\mu}$ and we know the generators of the ideals $I_{\mu'}$.

Let us define the spaces $V_{i,p}$. They are defined for $0 \le i \le \min(p, n-p)$. The space $V_{i,p}$ is contained in the span of $p \times p$ minors of $X$ (which we identify with $\wedge^p E \otimes \wedge^p E^*$). The subspace $V_{i,p}$ is spanned by the image of the map

$$\wedge^i E \otimes \wedge^i E^* \stackrel{1\otimes tr^{(p-i)}}{\longrightarrow} \wedge^i E \otimes \wedge^i E^* \otimes \wedge^{p-i} E \otimes \wedge^{p-i} E^* \longrightarrow \wedge^p E \otimes \wedge^p E^*$$

where the last map is a tensor product of exterior multiplication. The space $V_{i,p}$ can be alternatively defined as a span of linear combinations

$$\sum_{|J|=p-i} X(P, J|Q, J)$$

for all subsets $P, Q$ of $[1, n]$ of cardinality $i$. Here $X(P|Q)$ denotes the minor of $X$ corresponding to rows indexed by $P$ and columns indexed by $Q$.

The main result of [27] is:

**Theorem 5.4.3.** *The ideal $J_\mu$ is generated by the spaces $V_{0,p}$ ($1 \le p \le n$) and by the spaces $V_{i,\mu(i)}$ ($1 \le i \le n$) where $\mu(i) := \mu_1 + \cdots + \mu_i - i + 1$.*

Let us look at the equations defining the algebra $B_{\mu'}$. The restrictions of the representations $V_{i,p}$ that vanish on $Y_\mu$ certainly map to zero in $B_{\mu'}$. The main point of our proof of Theorem 5.4.3 is that one can prove using only combinatorial means to show that the restrictions of the representations $V_{i,p}$ described in Theorem 5.4.3 generate the ideal $I_{\mu'}$. More precisely, we will use the generators of $I_{\mu'}$ of (cf. [4]) described below in Proposition 5.4.4 (note that our $\mu$ is $\mu'$ in [4]).

Let $S = \{j_1, \ldots, j_k\}$ be a subset of $[1, n]$. For every $r$, $1 \le r \le \#S$, we define $e_r(S)$ to be the $r$-th elementary symmetric function in the variables $y_{j_1}, \ldots, y_{j_k}$.

**Proposition 5.4.4.** *The ideal $I_{\mu'}$ is generated by the following set $\mathcal{C}_\mu$ of symmetric functions on subsets of variables $y_1, \ldots, y_n$.*

$$\mathcal{C}_\mu = \{e_r(S) \mid k \ge r > k - d_k(\mu),\ \#S = k,\ S \subset [1, n]\}$$

*where $d_k(\mu) = \mu_{n-k+1} + \cdots + \mu_n$.*

**Proposition 5.4.5.** *The algebra $B_{\mu'}$ is the factor of $B$ by the ideal $I_{\mu'}$ generated by the restrictions of the representations $V_{i,\mu(i)}$ where $\mu(i) = \mu_1 + \ldots + \mu_i - i + 1$ (which are zero if $i > min(\mu(i), n - \mu(i))$) and by the elementary symmetric functions $e_j(y_1, \ldots, y_n)$.*

*Proof.* We compare the restrictions of representations $V_{i,p}$ to the generators exhibited in Proposition 5.4.4. The restriction of $V_{i,p}$ is the span of elements of the form $y_{j_1}\ldots y_{j_i}e_{p-i}(y_{j_{i+1}},\ldots,y_{j_p})$. where $j_1,\ldots,j_p$ are distinct elements of $[1,n]$.

Let us perform induction on $i$. For $i=0$, we get all elementary symmetric functions in $y_1,\ldots,y_n$ which can generate all elements of $\mathcal{C}_\mu$ with $\#S=n$. For $i=1$, we get all the symmetrizations of the functions $y_1e_{p-1}(y_2,\ldots,y_n)$. Modulo elementary symmetric functions we can, however, write

$$y_1e_{p-1}(y_2,\ldots,y_n)\equiv -e_p(y_2,\ldots,y_n)$$

because

$$e_p(y_1,\ldots,y_n)=y_1e_{p-1}(y_2,\ldots,y_n)\equiv e_p(y_2,\ldots,y_n).$$

This gives us the functions in $\mathcal{C}_\mu$ corresponding to $\#S=n-1$, since the condition $p>n-1-d_{n-1}(\mu)$ means exactly $p>\mu_1-1$.
For $i=2$ we get the symmetrizations of the functions $y_1y_2e_{p-2}(y_3,\ldots,y_n)$. But modulo the elements which are restrictions of those from $V_{1,p}$ and $V_{0,p}$ we have

$$y_1y_2e_{p-2}(y_3,\ldots,y_n)\equiv -e_p(y_3,\ldots,y_n)$$

because $e_p(y_1,\ldots,y_n)$ can be written as

$$e_p(y_3,\ldots,y_n)+y_1e_{p-1}(y_3,\ldots,y_n)+y_2e_{p-1}(y_3,\ldots,y_n)+y_1y_2e_{p-2}(y_3,\ldots,y_n)$$

The condition $r>n-2-d_{n-2}(\mu)$ means exactly $p>\mu_1+\mu_2-2$ so we generate in this way all elements in $\mathcal{C}_\mu$ with $\#S=n-2$.

Continuing in this way we prove that $\mathcal{C}_\mu$ is contained in the ideal generated by the restrictions of the representations $V_{i,p}$ listed in (8.2.5) of [28] which proves the proposition. $\square$

*Proof of Theorem 5.4.3.* First of all we need to know that the geometric method of calculating syzygies applied to a Springer type desingularization of $\mathcal{N}_\mu$ allows to show that the ideal $J_\mu$ is generated by the representations of type $S_{(1^i,0^{n-2i},(-1)^i)}E$ for $0\le i\le \frac{n}{2}$. This is Corollary (8.1.7) in [28].

We investigate the restriction map $\psi_\mu$ on the isotypic spaces corresponding to weights $\alpha(i)=(1^i,0^{n-2i},(-1)^i)$, $\lambda(i)=(2^i,1^{n-2i})$. Consider the restriction map

$$\psi_\mu^{\alpha(i)}:A_\mu^{\alpha(i)}\to B_{\mu'}^{\lambda(i)}$$

This map is an isomorphism. Consider the diagram

$$\begin{array}{ccc} A^{\alpha(i)}_{(n)} & \xrightarrow{\psi^{\alpha(i)}_{(n)}} & B^{\lambda(i)}_{(1^n)} \\ \downarrow & & \downarrow \\ A^{\alpha(i)}_{\mu} & \xrightarrow{\psi^{\alpha(i)}_{\mu}} & B^{\lambda(i)}_{\mu'} \end{array}$$

where the horizontal maps are the restriction isomorphisms and the vertical maps are natural surjections.

Denote by $J'_\mu$ the ideal generated by the representations defined in Theorem 5.4.3. By [28], Lemma 8.2.1, we have that $J'_\mu \subset J_\mu$.

Assume that there is a representation $S_{\alpha(i)}E$ in $A^{\alpha(i)}_{(n)}$ which is not in the ideal $J'_\mu$. Take such a representation of minimal possible degree $d$. It is therefore among the generators of $J_\mu$. It maps down to zero in $A^{\alpha(i)}_{\mu}$ and therefore its image $\Sigma^{\lambda(i)}$ in $B^{\lambda(i)}_{(n)}$ is generated by elements of lower degree in $I_{\mu'}$. Therefore there is a representation $\Sigma^\nu$ in degree $d-1$ in $I_{\mu'}$ for which $\Sigma^{\lambda(i)}$ is contained in the product $(\Sigma^\nu)S^1_{\mu'}$.

This however means by Lemma 5.4.2 that $S_{\alpha(i)}E$ is in the product $(S_\beta E)A^1_\mu$ where $S_\beta E$ is the representation restricting to $\Sigma^\nu$. Since $S_\beta E$ is in $J_\mu$, we get a representation in lower degree in $J_\mu$ but not $J'_\mu$. Since $J_\mu$ is generated by representations of weights $\lambda(i)$, we can get in lower degree a representation of a weight $\lambda(j)$ which is in $J_\mu$ but not $J'_\mu$. This gives a contradiction to the minimality of $d$.

**Remark 5.4.6.** There are fascinating combinatorial expressions for the polynomials $\tilde{K}_{\lambda,\mu}(q)$ due to Lascoux and Schützenberger (cf. [15]). They define a statistic, the *charge*, on the set $\mathrm{ST}(\lambda)_\mu$ of standard tableaux of shape $\lambda$ and of content $\mu$ (i.e. containing $\mu_1$ 1's, $\mu_2$ 2's, and so on) such that:

$$\tilde{K}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{ST}(\lambda)_\mu} q^{\mathrm{charge}(T)}$$

A general conjecture, giving the combinatorial description of $P_{\alpha,\mu}(q)$ for arbitrary $\alpha$, is described in [26] and proved for special partitions $\mu$.

## References


[1] E. Date, M. Jimbo, A. Kuniba, T. Miwa, M. Okado, *Paths, Maya diagrams and representations*, Adv. Stud. Pure Math. 19 (1989).

[2] C. De Concini and C. Procesi, *Symmetric functions, conjugacy classes and the flag variety*, Inv. Math. 64 (1981), no. 2, 203-219.

[3] S. Donkin, *The normality of closures of conjugacy classes of matrices*, Inv. Math. 101 (1990), no. 3, 717-736.

[4] A. Garsia and C. Procesi, *On certain $S_n$-modules and the q-Kostka polynomials*, Adv. Math. 94 (1992), 82-138.
[5] R. K. Gupta, *Generalized exponents via Hall-Littlewood symmetric functions*, Bull. Amer. Math. Soc., 16 (1987), 287-291.
[6] W. Hesselink, *Characters of the nullcone*, Math. Ann. 252 (1980), 179-182.
[7] W. V. D. Hodge, *Some enumerative results in the theory of forms*, Proc. Camb. Phil. Soc. 39 (1943), 22–30.
[8] W. V. D. Hodge, D. Pedoe, *Methods of Algebraic Geometry*, vols. I and II, Cambridge University Press, 1953.
[9] V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press.
[10] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. 85 (1963), 327-404.
[11] H. Kraft, *Conjugacy classes and Weyl group representations*, Young taleaux and Schur functors in algebra and geometry, Asterisque 87-88, Soc. Math. France, Paris (1981), 191-205.
[12] V. Kreiman, V. Lakshmibai, P. Magyar, J. Weyman, *Standard bases for affine $SL_n$-modules* (submitted to IMRN).
[13] S. Kumar, *Infinite Grassmannians and moduli spaces of G-bundles*, Vector Bundles on Curves, Springer LNM 1649 (1997), 1-50.
[14] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Birkhäuser, Progress in Mathematics 204 (2002).
[15] A. Lascoux and M.P. Schützenberger, *Sur une conjecture de H.O. Foulkes*, C.R. Acad. Sci. Paris 286A (1978), 323-324.
[16] P. Littelmann, *Contracting modules and standard monomial theory*, J.A.M.S. 11 (1998), 551–567.
[17] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981) 169–178.
[18] G. Lusztig, *Canonical bases arising from quantized universal enveloping algebras*, J.A.M.S. **3** (1990) 447–498.
[19] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Math. Monographs (1979).
[20] P. Magyar, *Affine Schubert Varieties and Circular Complexes*, math.AG/0210151 (2002).
[21] P. Magyar, *Littelmann Paths for the Basic Representation of an Affine Lie Algebra*, math.RT/0308156 (2003).
[22] V. B. Mehta and W. van der Kallen, *A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices*, Compositio Math. 84 (1992), no. 2, 211–221.
[23] C. Musili, *Postulation formula for Schubert varieties*, J. Indian Math. Soc., 36 (1972), 143-171.
[24] A. Pressley and G. Segal, *Loop Groups*, Oxford Science Publications, Clarendon Press, Oxford (1986).
[25] M. Shimozono, *Affine type A crystal structures on tensor products of rectangles, demazure characters and nilpotent varieties*, math.QA/9804039 (1998).
[26] M. Shimozono and J. Weyman, *Bases for Coordinate Rings of Conjugacy Classes of Nilpotent Matrices*, J. Alg. 220, no. 1 (1999), 1-55.
[27] J. Weyman,*The equations of conjugacy classes of nilpotent matrices*, Invent. Math. 98(1989), 229-245.

[28] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics Vol. 149, Cambridge University Press, 2003.

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